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OF MATHEMATICAL SCIENCE

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THE TWENTY-NINTH ANNUAL MEETING OF THE SOCIETY

The twenty-ninth annual meeting of the Society was held at Harvard University on Wednesday and Thursday, December 27–28, 1922, during the period of the Convocation week meetings of the American Association for the Advancement of Science and immediately preceding the meeting of the Mathematical Association of America. Dormitory accommodations were furnished by Harvard University and Radcliffe College, and Phillips Brooks House was used as a social center for the mathematical group. Not only was there a record attendance at the meeting, but members found it stimulating and pleasant to come into contact with scientists working in fields related to mathematics. At the close of the sessions it was voted to express the thanks of the Society to Harvard University for its hospitality and to the departments of mathematics at both Harvard and the Massachusetts Institute of Technology for the admirable way in which arrangements for the meetings were carried out.

Members of the Society were invited to hear the address of Professor E. H. Moore, retiring president of the American Association for the Advancement of Science, on *What is a number system?*, delivered Tuesday evening at the Massachusetts Institute of Technology. Joint sessions were held on Wednesday afternoon with Section A of the American Association for the Advancement of Science, at the Massachusetts Institute of Technology, and on Thursday afternoon with the Mathematical Association of America and Section A at Harvard University. An enjoyable occasion was the joint dinner of the three organizations held at the Massachusetts Institute of Technology on Thursday evening.

The attendance included the following 148 members of the Society:

Alexander, E. S. Allen, Archibald, Atchison, Bailey, Beatley, W. J. Berry, Bill, Birkhoff, Bliss, Borden, Bowler, B. H. Brown, E. P. Brown, Bryan, M. Buchanan, W. G. Bullard, R. W. Burgess, Burkett, Burwell,

Cairns, Cajori, B. H. Camp, Carr, B. E. Carter, Clawson, Coates, Coolidge, Copeland, Crum, Currier, Denton, Dimick, Dodd, Eisenhart, L. C. Emmons, Esty, G. C. Evans, G. W. Evans, Falconer, Floyd Field, Fields, C. H. Forsyth, Fort, Philip Franklin, Frizell, Garabedian, Gerst, Gilman, Gleason, Glenn, Glover, Gould, M. C. Graustein, W. C. Graustein, Gummer, Hammond, J. G. Hardy, Haskins, Hazlett, Hicks, Hille, Hodgkins, Huntington, Hurwitz, Ingraham, Kazarinoff, Kellogg, Kennelly, W. D. Lambert, Larew, Leib, Leyzerah, Lipka, McGavock, MacDuffee, MacPherson, H. P. Manning, Marriott, G. A. Miller, Molina, C. L. E. Moore, C. N. Moore, E. H. Moore, R. K. Morley, H. C. M. Morse, F. R. Moulton, F. H. Murray, Musselman, Newkirk, Olds, Osgood, Packer, Pattillo, Pell, Perkins, Phillips, Pitcher, Ransom, Ranum, L. H. Rice, R. G. D. Richardsons, H. L. Rietz, E. D. Roe, Roever, Ruger, Rutledge, Safford, Seely, Sharpe, Shaub, Shewhart, Silverman, W. G. Simon, Sinclair, Slaught, Slobin, C. E. Smith, D. E. Smith, S. E. Smith, M. J. Sperry, Stetson, Swartzel, J. S. Taylor, J. M. Thomas, Tracey, Tyler, Van Vleck, Veblen, Vivian, Waddell, J. L. Walsh, Webster, Wedderburn, Weisner, M. E. Wells, H. S. White, D. E. Whitford, Whittemore, Wiener, C. E. Wilder, F. B. Williams, Willis, R. G. Wood, F. S. Woods, J. M. Young, J. W. Young, Zeldin.

President Bliss presided at the session Wednesday morning, relieved by Professor Veblen. President Veblen presided at the joint session on Wednesday afternoon, Professor Osgood on Thursday morning, and Professor G. A. Miller, Vice-President of Section A, at the joint session on Thursday afternoon.

At the meeting of the Council, the following twenty-two persons were elected to membership in the Society:

Professor Ralph Beatley, Harvard Graduate School of Education;
Mrs. Ethelwynn Rice Beckwith, Harvard University;
Rev. Dr. Martin Brennan, St. Louis, Mo.;
Professor James Bruce Coleman, University of South Carolina;
Professor Marion Alonzo Eason, United States Naval Academy;
Mr. Hervey Crandall Hicks, Brown University;
Mr. William Charles Janes, Kansas State Agricultural College;
Mr. Carl Otto Lampland, Lowell Observatory;
Mr. Frank Solimena Langellotti, New York, N. Y.;
Miss Martha Peirce McGavock, Wellesley College;
Mr. Albert Eugene Meder, Columbia University;
Dr. Winifred Edgerton Merrill, Mamaroneck, N. Y.;
Dr. Harvey Pierson Pettit, University of Illinois;
Mr. Pincus Schub, Hebrew School of Newark;
Professor Charles Cutler Spooner, Northern State Normal School, Marquette, Mich.;
Mr. Charles Aynes Stanwick, Western Electric Company;

Professor Tsurusaburo Takasu, Tôhoku Imperial University;
Dr. Louis Ten Eyck Thompson, Kalamazoo College;
Dr. Elihu Thomson, General Electric Company;
Professor Edward Lee Thorndike, Teachers College, Columbia University;
Miss Virginia Watts, New Orleans, La.;
Mr. Frederick W. Weston, Wynnewood, Pa.

Twenty-five applications for membership were received. The Secretary announced that the following members of the London Mathematical Society had entered the American Mathematical Society in consequence of the reciprocity agreement recently arranged between the two Societies:

Professor Mathew J. Conran, University College, Cork;
Mr. Krishna Prasad Dè, University College, Rangoon;
Professor Arthur Lee Dixon, Magdalen College, Oxford;
Principal John Leigh Smeathman Hatton, East London College, University of London;
Professor Harold Hilton, Bedford College, University of London;
Sir Joseph Larmor, St. John's College, Cambridge;
Professor George Henry Livens, University College, Cardiff;
Professor William Proctor Milne, University of Leeds;
Professor Archibald Read Richardson, University College, Swansea;
C. M. Ross, King's College, Cambridge;
Mr. Reginald Owen Street, University of Liverpool;
Mr. Bertram Martin Wilson, University of Liverpool.

The Council announced that the invitation of Vassar College to hold the summer meeting of 1923 at that institution had been accepted and that the annual meeting would be held in New York City.

The committee on membership made a final report through its chairman, President Bliss. Since October 1921 the committee by enlisting the cooperation of several members of the Society has obtained some 228 new members, 33 new subscriptions to the TRANSACTIONS and $372.00 in gifts from members to the regular funds. It recommended that in view of the still critical state of the finances of the Society a new committee on membership be appointed. In discharging the committee, the Council thanked the chairman for the splendid service rendered to the Society. A new committee on membership was appointed consisting of Professors Abraham Cohen, W. H. Roever, and C. E. Smith (permanent secretary).

Professor G. A. Bliss was elected a member of the International Mathematical Union to fill the vacancy caused by Professor Veblen's succession to the presidency of the Society. Professors C. N. Moore and R. G. D. Richardson were appointed to represent the Society on the Council of the American Association for the Advancement of Science for the year 1923.

The Council authorized the appointment by the President of standing committees on endowment, and on policy and budget.

The total membership of the Society is now 1127, including 82 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 538; the number of papers read was 228. The number of members attending at least one meeting was 377. At the annual election 246 votes were cast.

The reports of the Treasurer and the Auditors (Mr. S. A. Joffe and Professor W. J. Berry) were received, showing a balance $13,232.36, exclusive of special funds; of this $5782.85 is reserved to secure the life memberships. It was explained that the large balance on hand is due to the fact that the Transactions of the Society is eighteen months late in appearing. Sales of the Society's publications during the year amounted to $3,298.43.

The Library now contains 6,203 volumes, exclusive of members' dissertations.

At the annual election, which closed on Wednesday morning, the following officers and other members of the Council were chosen:

President, Professor OSWALD VEBLEN.

Vice-Presidents, Professor R. L. MOORE, Professor H. W. TYLER.

Secretary, Professor R. G. D. RICHARDSON.

Treasurer, Professor W. B. FITE.

Librarian, Professor R. C. ARCHIBALD.

Committee of Publication

Professors E. R. HEDRICK, W. A. HURWITZ, J. W. YOUNG.

Members of the Council to serve until December, 1925

Professor E. T. BELL, Professor J. C. FIELDS,
Professor W. H. BUSSEY, Professor ARTHUR RANUM.

The amendment to the constitution designating as permanent member of the Council any ex-Secretary who had served ten years was carried.

The joint session on Wednesday afternoon was devoted to a symposium on *Space and Time* and was attended by about three hundred scientists. The following papers were read:

I. *The logic of space and time*, by Professor G. D. Birkhoff.

II. *The physical meaning of space and time*, by Professor P. W. Bridgman.

III. *The astronomical measures of space and time*, by Professor Harlow Shapley.

The program of the joint session on Thursday afternoon was as follows:

I. *The reduction of singularities of plane curves by birational transformation*, by Professor G. A. Bliss, retiring President of the Society. This paper will appear in full in the April number of this BULLETIN.

II. *The grafting of the theory of limits on the calculus of Leibniz*, by Professor Florian Cajori. (Address delivered at the request of the Mathematical Association of America.)

III. *Geometry and physics*, by Professor Oswald Veblen, retiring Vice-President of Section A.

Titles and abstracts of the papers read at the regular sessions of the Society follow below. Dr. Bray's paper was read by Professor Evans. The papers of Professors Graustein, C. L. E. Moore, Craig, Cajori, Bell, R. L. Moore, and Camp, Dr. Post, Professor Schwatt, Mr. Poritsky, Professors Hitchcock and Cowley, Mr. Murray, Professor Alexander, and Mr. Raynor, and the second papers of Dr. Walsh, Dr. B. H. Brown, Dr. Wiener, Dr. Franklin, Professor Coolidge, Professor Miller, and Dr. MacDuffee were read by title. Mr. Brinkmann and Mr. Linfield were introduced by Professor Birkhoff and Professor Hitchcock by Professor C. L. E. Moore.

1. Dr. J. L. Walsh: *On the location of the roots of polynomials.*

It is the purpose of this paper to prove the following theorem: Let circles C_1, C_2, $\cdots$, C_k, whose centers are the points α_1, α_2, $\cdots$, α_k, be the loci of n_1, n_2, $\cdots$, n_k roots of a polynomial $f(z)$ which has no other roots, where the circles C_i are all equal and their centers α_i all lie on a line parallel to the axis of reals. If these circles are sufficiently small, and if the polynomial

$$a_0z^n + na_1z^{n-1} + n(n-1)a_2z^{n-2} + \cdots + n(n-1)\cdots 2\cdot 1a_n, \qquad (n = n_1 + n_2 + \cdots + n_k),$$

has only real roots, then the locus of the roots of the polynomial $F(z) = a_0f(z) + a_1f'(z) + a_2f''(z) + \cdots + a_nf^{(n)}(z)$ consists of the circles C_j' that are equal to the circles C_i and whose centers are the roots of $F(z)$ when the roots of $f(z)$ are the points α_1, α_2, $\cdots$, α_k of multiplicities n_1, n_2, $\cdots$, n_k respectively. Any circle C_j' which has no point in common with any of the other circles C_m' contains a number of roots of $F(z)$ equal to the multiplicity of its center as a root of $F(z)$ when the roots of $f(z)$ are the points α_i.

2. Dr. J. L. Walsh: *A generalization of normal congruences of circles.*

This paper appeared in the December BULLETIN.

3. Dr. Eleanor P. Brown: *Expansion theorems for a certain homogeneous integral equation.*

This paper deals with a Fredholm linear homogeneous integral equation $u(x) = -\rho^2\int_\alpha^\beta K(x, \xi)u(\xi)d\xi$, whose kernel is such that, though continuous and possessing continuous derivatives on either side of the line $x = \xi$, in the x, ξ plane, it has certain "finite jumps" in its derivatives as the point (x, ξ) crosses this line. Provided certain orthogonality relations are not satisfied, the equation can be reduced to a differential equation of Sturm-Liouville type with the addition of certain integral terms, along with a pair of boundary conditions. These integral terms have little effect on the solutions of the equation, and it is shown that the characteristic numbers and functions behave essentially like those of the corresponding Sturm-Liouville system. The fundamental theorem is proved that if we set up the formal Fourier series, with respect to each of the two above-mentioned systems, of any function $\Phi(x)$ which, in absolute value, is integrable, then the difference

(taken termwise) between these two series converges absolutely and uniformly to zero.

4. Dr. B. H. Brown: *The equilong transformations of euclidean space.*

The equations of the equilong transformations of euclidean space, in Bonnet oriented tangential coordinates, are

Direct: $U = U(u),\ V = V(v),\ W = \sqrt{U'V'}\,w + F(u, v);$

Indirect: $U = U(v),\ V = V(u),\ W = \sqrt{U'V'}\,w + F(u, v).$

The forms given by Blaschke (ARCHIV, 1910), and all subsequent references in the literature contain a $\pm$ sign under the radicals, which is incorrect.

The main portion of this paper deals with the invariance of differential expressions and equations under these, or subgroups of these transformations. Applications are made to one-sided surfaces, to minimal surfaces and their associated surfaces, and to a large class of surfaces defined by second order linear partial differential equations, including the surfaces of Appell, of Goursat, and of Bonnet. The following theorem is proved: Any oriented non-developable surface may, by each of two and only two distinct, direct, equilong transformations, be transformed into any other oriented non-developable surface, and that with an arbitrary analytic directly conformal mapping of their spherical representations. This theorem was suggested by Study, but he stated, incorrectly, that there was one and only one such transformation.

5. Dr. B. H. Brown: *Contact transformations linear in x, y, and z; applications to equilong transformations.*

A transformation of Lie planar elements, in which X, Y, and Z are linear in x, y, and z, carries planes into planes if and only if all the three-rowed minors of a certain ten-by-three matrix vanish. Necessary and sufficient conditions for equilong transformations, a subgroup of the preceding group, follow easily. For a particular subgroup of the equilong group, these conditions reduce to two partial differential equations which are necessary and sufficient conditions that the parameters u and v of a general surface $x = x(u, v)$ have the geometric significance of p and q. Certain rational solutions of these equations furnish the parametric equations of well known one-sided surfaces. The transformation of Bonnet is shown to be equilong.

6. Professor W. C. Graustein: *On the determination of a surface by its principal normal curvatures and spherical representation.*

It is shown in this paper that a non-developable surface is, in general, uniquely determined, to within its position in space, by the linear element of its spherical representation and its principal normal curvatures, and that the exceptional cases, for which definitive conditions are given, lead either to two or to a single infinity of non-congruent surfaces.

7. Professor W. C. Graustein and Mr. B. O. Koopman: *A necessary and sufficient condition that a map of two surfaces be isometric.*

It is well known that in order that a map of two surfaces be isometric it is necessary that it be geodesic and that the total curvature at corresponding points be equal; it is also known that neither of these conditions is alone sufficient. This paper shows that these two conditions taken together are sufficient, except when the two surfaces are of constant total curvature.

8. Professor C. L. E. Moore: *Deformation of surfaces in 4-space.*

The author derives characteristic equations which must be satisfied if two surfaces correspond by orthogonality of elements, and shows the connection with infinitesimal deformation.

9. Professor Joseph Lipka: *On conformal parallelism.*

We consider two curved spaces V_n and V_n', one of which is a conformal representation of the other. Two linear elements in V_n are conformally parallel when their conformal representations in V_n' are parallel in the sense of Levi-Civita. The author proves a number of theorems on conformal parallelism in V_n analogous to the theorems on parallelism in V_n'. A natural family of curves in V_n plays the rôle of the geodesics in V_n'. In particular, a study is made of the motion of a linear element by conformal parallelism around an infinitesimal cycle, and of the motion of a linear element both by parallelism and conformal parallelism along a curve.

10. Professor Joseph Lipka: *On irreversible systems in dynamics.*

An irreversible field of force is defined as one in which the Lagrangian or kinetic potential has the form

$$L = L_2 + L_1 + L_0 = \tfrac{1}{2}\sum_{ik} a_{ik}\frac{dx_i}{dt}\frac{dx_k}{dt} + \sum_i \phi_i \frac{dx_i}{dt} + U,$$

where t is the time, and a_{ik}, ϕ_i, and U are functions of the coordinates only. In this paper, the author proves the theorem that if in an irreversible field of force with a given constant of energy, particles are projected from all points of an arbitrary surface in a direction such that the vector sum of the velocity and the vector ϕ_i is normal to the surface, points which they reach with equal actions (action $= \int (2L_2 + L_1)dt$) lie on a surface also cutting the paths so that the vector sum of the velocity and the vector ϕ_i is normal to the surface. (If the vector ϕ_i vanishes, there is a reversible field, and the theorem reduces to the well known theorem of Thomson and Tait.) By means of Hamilton's canonical equations and the theory of contact transformations, this theorem and its converse are proved directly for curved spaces of any dimensionality.

11. Dr. J. S. Taylor: *A 4-space representation of complex plane analytics.*

Kwietniewski and K. Kommerell have set up a one-to-one correspondence between a certain family of surfaces in ordinary 4-space S_4 and the curves (real and imaginary) of plane analytics. In the present paper the author investigates the problem of representing plane analytics as a whole in a modified 4-space Σ_4, extending for use in the 4-space the complex measures of distance and angle of complex plane analytics. The "rigid motions" of S_4 and of Σ_4 are discussed, both in relation to each other and in relation to the transformations effected on equations defining regular functions of a complex variable. Among other subgroups, the group of rotations in S_4 is obtained under which all surfaces representing regular functions remain surfaces representing regular functions. Incidentally, a family of 4-space rotations is given that has apparently been overlooked. The author makes some applications to the theory of functions of a complex variable.

12. Dr. Norbert Wiener: *Differential-space.*

The author shows that procedure to the limit in questions relating to the Brownian movement gives rise to a space of infinitely many dimensions of a character fundamentally distinct from the familiar function-space discussed by such writers as Fréchet, Volterra, and Lévy, and employs the methods of Lévy in developing integration in this space.

He discusses the introduction of orthogonal coordinates in differential-space, and develops a complete set of normal and orthogonal functionals, related to the Hermite polynomials.

13. Dr. Norbert Wiener: *Note on the series* $\Sigma(\pm 1/n)$.

The author shows that with an arbitrary haphazard selection of signs for the terms of $\Sigma(\pm 1/n)$, it is of zero probability that it diverge.

14. Professor J. L. Coolidge: *On the existence of curves with assigned singularities.*

This paper appeared in the December BULLETIN.

15. Professor J. L. Coolidge: *The Gaussian law of error for any number of variables.*

In this paper the Gaussian law of error is developed for any number of variables, by a natural extension of the classical method, and the coefficients are determined in terms of individual mean errors and correlation coefficients.

16. Mr. H. W. Brinkmann: *On conformal Riemann spaces.*

The author proves the following theorem: An n-dimensional Riemann space conformal to euclidean space can be considered as the intersection of a null hypersphere and some other hypersurface in $(n+2)$-dimensional euclidean space. By means of this theorem Schouten's necessary and sufficient condition that a Riemann space be conform-euclidean is readily derived. The method of proof is entirely elementary and can be extended to prove a similar theorem for any two conformal Riemann spaces. Theorems concerning Riemann spaces containing certain sets of n-fold orthogonal systems of hypersurfaces are also obtained by this method.

17. Professor E. S. Allen: *On certain power series with positive coefficients.*

Each coefficient of a series of the type considered is a function of all preceding coefficients. Information is obtained concerning the interval of convergence, in some cases through the expression of the coefficients in closed form.

18. Dr. Philip Franklin: *A qualitative definition of the trigonometric and hyperbolic functions.*

This paper appeared in the February BULLETIN.

19. Dr. Philip Franklin: *Tensors of given types in Riemann space.*

This paper begins by defining an arithmetic tensor in euclidean space as a tensor whose components are constant, independent of both the position of the point in question and

the cartesian coordinates used, and proving some of its properties. These properties are then extended to tensors in Riemann space, the final theorem being: The most general tensor of given even order involving the g_{ij}'s and their first and second derivatives, these last linearly, is a linear combination of tensors which may be formed from the curvature tensor R^{i}_{fgh} and the fundamental quadratic tensor g_{ij} by multiplication and contraction. There are no such tensors of odd order.

These theorems enable us to set up many of the equations used in physics by taking the most general tensor of the type wanted, and after determining the constants by special considerations equating it to zero. The paper has been accepted for publication by the PHILOSOPHICAL MAGAZINE.

20. Mr. B. Z. Linfield: *A contribution to particle geometry.*

In this paper the varieties of a universe of particles are classified according to the nature of the subvariety which is embedded in a variety and which is in the neighborhood of an arbitrary element (particle) of this same variety. After considering the n-dimensional, normal varieties of the second order, S_n^2, the author concludes with the following theorem: There exists a regular map of regions whose boundaries are convex which is equivalent to an arbitrary regular map.

21. Dr. H. E. Bray: *Proof of a formula for an area.*

This paper will appear in full in this BULLETIN.

22. Professor C. F. Craig: *On the Riemann zeta function.*

In this note a functional relation for the Riemann zeta function is established and its relation to the well known formula given by Riemann is pointed out. It is believed that the relation is new.

23. Professor Florian Cajori: *Mathematical signs of equality.*

Recorde's sign = of 1557 did not again appear in print until 1618, but after 1631 it gained a firm foothold in England. Certain Continental writers used = in the seventeenth century in four different senses. Seven different signs of equality were proposed, but Descartes' $\propto$ was the only serious rival, which about 1680 threatened to displace = on the Continent. The two inventors of the calculus, Leibniz and Newton, favored = and apparently brought about its final victory.

24. Professor E. T. Bell: *Extended class number relations.*

A sum of the form $\Sigma X(n - \sigma^2)$ in which X is any class number function and Σ extends to certain of the integers σ

rendering $n - \sigma^2 > 0$ is here called principal of order c and degree μ if each σ/c is the μth power of an integer. A relation involving a principal sum of order c and degree μ is called extended of rank λ and range (A, B), where $A < B$, if the relation involves also the arbitrary even function f having as arguments only integers whose λth powers lie between A and B. By a simple new method a closely interconnected set of 29 such extended relations is found in which c is an arbitrary constant integer $\neq 0$, μ is an arbitrary odd constant integer > 0, λ is an arbitrary even constant integer > 0, and A, B $(A < B)$ are arbitrary constants. Hitherto only instances of the cases $c = 1, 4$, or a prime, $\mu = 1$, $\lambda = 2$, $A = 0$, $B = \infty$, $f(x) = 1$ for all values of x have appeared. For these special values the extensions degenerate to known relations of Kronecker, Liouville, and Humbert, or to others of similar types.

25. Professor R. L. Moore: *Concerning the uniform convergence of certain sequences of equicontinuous arcs.*

Suppose that (1) AB and CD are two parallel lines, (2) for every n, A_n and C_n are points lying on AB and CD respectively and A_nC_n is a simple continuous arc which, except for its end points, lies between AB and CD, (3) the sequence of arcs $A_1B_1, A_2B_2, \cdots$ is equicontinuous and has, as its limiting set, an arc AC which lies, except for its end points, entirely between AB and CD. Then there exists, for each n, a one-to-one continuous correspondence, T_n, between AC and A_nC_n such that the set of correspondences $T_1, T_2, T_3, \cdots$ is equicontinuous and such that, *with respect to this set of correspondences, the sequence* A_1C_1, A_2C_2 *converges uniformly to* AC, that is to say, it is true that for every positive number ϵ there exists a positive number δ_ϵ such that if $n > \delta_\epsilon$ then, for every point P of AC, the distance from P to $T_n(P)$ is less than ϵ.

It is shown that a sequence of arcs $A_1C_1, A_2C_2, \cdots$ is not necessarily equicontinuous* even though there exist two equicontinuous sequences of functions $f_1(t), f_2(t), \cdots$ and $F_1(t), F_2(t), \cdots$ such that, for every n, A_nC_n is represented parametrically by the pair of equations $x = f_n(t)$ and $y = F_n(t)$ $(0 \leqq t \leqq 1)$.

26. Professor R. W. Burgess: *A non-linear partial regression equation.*

In the theory of partial regression equations, it is assumed that each variable is a linear function of the others. This assumption implies, in the case of three variables, that the coefficient of regression of one variable on a second is constant

*Cf. TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), p. 42.

no matter what values are assigned to the third variable. As in certain statistical problems this seems contrary to the essential nature of the relationship between the variables, the author suggests the use of the equation $x_1 = ax_2x_3 + bx_2 + cx_3 + d$, and has determined the best values of the coefficients a, b, c, d, in the sense of the method of least squares. He has applied his method to a further analysis of H. L. Moore's problem of predicting the yield of cotton from the May rainfall and June temperature, and finds that the new method results in an improvement in accuracy of prediction.

27. Professor B. H. Camp: *On a short method of least squares.*

It is known that, when the independent variables of a set of observational equations are equi-spaced, certain simplifications are available which greatly shorten the least square solution (cf. H. M. Roeser, SCIENTIFIC PAPERS OF THE BUREAU OF STANDARDS, No. 388). It is shown by the author that these may frequently be employed, with negligible error, when the independent variables are only approximately equi-spaced. Expressions for the error are derived which may be computed quickly, after the short solution has been obtained, and which also permit the computer to estimate roughly, in advance of his computation, the size of the error to be expected. This paper will appear in the ANNALS OF MATHEMATICS.

28. Professor Olive C. Hazlett: *Finiteness theorems for formal modular covariants.*

By the aid of a symbolic notation explained in a previous paper, the author proves that all formal modular covariants of a system of binary forms with respect to the Galois field $GF[p^n]$, of order p^n, are congruent, in the field, to polynomials in a finite number of such covariants.

29. Professor R. W. Marriott: *Determination of all types of subgroups of the group of order p^4, where p is a prime.*

The exact number of every type of subgroup of order p^α, $\alpha < 4$, has been determined, in this paper, for every group of order p^4, and these subgroups expressed in terms of the generating operations of the group of order p^4. The following theorem was made use of: Let Gp^m be a group of order p^m of a certain given type, and Gp^α a group of order p^α of specified type, $\alpha < m$. Let N be the number of ways operations can be selected from Gp^m which satisfy the defining relations of Gp^α, but no relations independent of these. Then the number of subgroups of Gp^m of order p^α and type Gp^α is equal to N divided by the order of the group of isomorphisms of Gp^α.

The author takes for generating elements of Gp^e the most general operations in the group of order p^m, subject these operations to the defining relations of Gp^e, and impresses on them the further condition that they shall obey no relations other than these defining relations of Gp^e. The method leads always to a set of congruences involving the exponents of the generators, the number of solutions of which gives the value of N.

30. Professor O. E. Glenn: *The sieve of Eratosthenes.*

The distribution of primes in the infinite sequence of positive integers is the subject of many researches dating from the work of Eratosthenes (276 B. C.). Tables of figurate numbers defined in the author's paper presented at the Chicago meeting, April, 1922, contain primes distributed practically at random. The method there explained for finding the figurate numbers which are divisible by a given prime p is equivalent to a generalization to two dimensions of Eratosthenes' sieve. We prove that for extensive classes of plexuses the complete description of divisibility by p may be obtained from a finite period parallelogram of residues recurrent to infinity within the plexus, and the method of the sieve gives general results. The developments of the paper connect with some ancient problems in numbers and the finished manuscript is to present extensive lists of primes from typical figurate number tables.

31. Professor G. A. Miller: *New applications of a fundamental theorem of substitution groups.*

The fundamental theorem considered in this paper may be stated as follows: If the cycle $a_1a_2\cdots a_l$ appears in at least one substitution of the group G and if we count this cycle and its conjugates under G as many times as each of them appears in different substitutions of G, then the total number of letters in these cycles is equal to the order of G. From this theorem one can readily obtain the following useful theorem: The number of the different sets of conjugate cycles under any substitution group of degree n is $n - k$, k being the number of its systems of intransitivity. In particular, all the possible transitive groups of degree n have in common the property that each of them contains the same number of sets of conjugate cycles. The author aims to exhibit the applications of these theorems by using them in a new determination of the substitution groups of degrees four and five, and in finding exponents to which numbers belong with respect to a modulus which is a power of a prime number.

32. Professor G. A. Miller: *New proof of the theorem that every prime power group contains invariant operators.*

The author shows that his proof of Sylow's theorem * can readily be extended to include a proof of the fundamental theorem that every group of order p^m (p a prime) contains at least one invariant operator of order p. To prove this extension it is convenient to let p^β represent the highest power of p which divides n instead of the highest power of p which does not exceed n, since in both cases the number of substitutions of degree p^β and order p in the symmetric group of degree n is prime to p. The regular group G whose order is divisible by p^β but not by $p^{\beta+1}$ contains a subgroup of order p^β whose constituents are regular, and the substitutions of the symmetric group of degree n which are of degree p^β and commutative with each substitution of this subgroup of order p^β must be found in the conjoints of these regular constituents. As the number of these conjoints is prime to p, each of them contains a number of substitutions of order p which is prime to p, and the subgroup of order p^β contains a number of substitutions of order p which is prime to p, and therefore an invariant operator of order p. It follows also the number of operators of order p in a group whose order is divisible by p is prime to p.

33. Dr. C. C. MacDuffee: *Covariants and transformable systems.*

This paper appeared in the January BULLETIN.

34. Dr. C. C. MacDuffee: *A theorem on covariants of forms with an application to linear algebras.*

It is proved that for every value of j the parameters α_{ij} of a linear homogeneous transformation replacing the system of forms f_k by the system f_k' are transformed by an arbitrary transformation cogrediently with the variables x_i. This theorem gives a means of calculating the rational integral covariant of the linear algebra in n units (one of which is a principal unit) which vanishes identically when and only when the right-hand (left-hand) rank of the algebra is less than n.

35. Professor C. N. Moore: *On the Cesàro summability of the double Fourier's series.*

The main result of this paper is the theorem that the double Fourier's series corresponding to any function $f(x, y)$ that has a Lebesgue integral in the region $(-\pi \leqq x \leqq \pi, -\pi$

* ANNALS OF MATHEMATICS, (2), vol. 16 (1915), p.169.

$\leqq y \leqq \pi$) will be summable (C, ρ, δ), where ρ and δ are any two positive constants, throughout the above region, except for a set of points of measure zero. This is the extension to double Fourier's series of a theorem with regard to ordinary Fourier's series due to G. H. Hardy.

36. Professor L. L. Silverman: *A certain type of regular transformation.*

In the study of regular transformations of the form $y_n = \sum_{k=1}^{n} a_{nk}x_k$, several authors have defined a_{nn} by means of a function $f(x)$ satisfying certain conditions in the interval (0, 1), and have then defined the numbers a_{nk}, $k < n$, by the condition that the given transformation be permutable with the special transformation $y = (1/n)\sum_{k=1}^{n} x_k$. These transformations, which we shall call *of type A*, include the definitions of Cesàro and Hölder for special choice of $f(x)$. The present author studies the transformations *of type B*, for which $a_{nn} = 1 - f((n-1)/n)$, $a_{nk} = f(k/n) - f((k-1)/n)$, $k < n$, where $f(x)$ satisfies the same conditions in the interval (0, 1) as in the case of transformations of type A. An objection to this new type of transformation is that for no choice of $f(x)$ does it represent any of the transformations of Cesàro or Hölder, except in the case of orders 0 and 1. On the other hand, if $f_r(x)$ corresponds to C_r, the transformation of Cesàro of order r, according to the plan of type A, then the transformation B_r, obtained from $f_r(x)$ according to the plan of type B, is proved to be more general than C_r, for each positive integer r greater than unity. The transformations of type B may be written in the form of an integral, which is identical with the formula for the transformations of functions of a continuous variable.

37. Professor H. B. Phillips: *Nets and the Dirichlet problem.*

The author considers the function defined on a rectangular net by the condition that its value at each node is equal to the average of its values at the adjacent nodes. By considering nets with progressively finer mesh he approximates to the solution of the Dirichlet problem in the plane.

38. Professor H. C. M. Morse: *When does the inverse problem of the calculus of variations lead to an integral of arc length?*

The general inverse problem of the calculus of variations is to find an integral for which a given two-parameter family of curves are the extremals in case the integral be minimized. Darboux has shown that the general problem has an infinite number of solutions. The present writer inquires what are

the conditions on the given family of curves that they may serve as extremals arising from an integral of arc length.

39. Mr. Louis Weisner: *Groups whose maximal cyclic subgroups are independent.*

A cyclic subgroup of a group is said to be a maximal cyclic subgroup of the group if it is not contained in any larger cyclic subgroup of the group. The present paper, which is the author's dissertation, concerns itself with groups whose maximal subgroups are independent; that is, groups in which no two maximal cyclic subgroups have an element in common besides identity. The cyclic group and the type $(1, 1, 1, \cdots)$ abelian groups are the only abelian groups having this property. Of prime power groups, the only ones are the cyclic, dihedral, the type $(1, 1, 1, \cdots)$ abelian group, and groups conformal with the type $(1, 1, 1, \cdots)$ abelian group. The only other solvable groups having the property are those which contain an invariant cyclic subgroup, and the octahedral group, which falls into a class by itself. All other such groups are insolvable and usually simple. General properties of such groups are also considered in the paper.

40. Dr. George Rutledge: *The polynomial determined by* $2n + 1$ *evenly spaced points.*

Examples of the polynomials whose general form the author discusses are the following: $(4 - 3x^2)/4$, $(52 - 51x^2 + 36x^4)/52$, $(3{,}472 - 3{,}468x^2 + 3{,}321x^4 - 2{,}187x^6)/3{,}472$, $(48{,}412 - 48{,}409x^2 + 48{,}204x^4 - 44{,}928x^6 + 27{,}648x^8)/48{,}412$, which agree in value with the function $1/(1 + x^2) = 1 - x^2 + x^4 - x^6 + \cdots$ for (respectively) three, five, seven, and nine equally spaced values of x on the interval $x = -1/\sqrt{3}$ to $x = +1/\sqrt{3}$, end values included. These polynomials are the basis of many well known interpolation formulas.

41. Dr. E. L. Post: *Visual intuition in spherical and elliptic space: Einstein's finite universe.*

The method of the present paper is almost the same as the one used in the author's paper on Lobachevskian intuition. However, the difficulties involved in the conception of the finiteness of both the spherical and elliptic spaces and the one-sidedness of the latter require further elucidation. This is accomplished by imagining a plenitude of the straight lines of these spaces, uniformly distributed in them, and noting just how they bind the spaces together; or by noting how the spaces

can be broken up into a finite number of adjacent but non-overlapping cells. In both cases the desired intuition is obtained by noting how these straight lines and cells vary with changes in the position of the observer. Since Einstein's finite universe is one of these types, the present paper may serve to dispel some popular misconceptions.

42. Dr. E. L. Post: *A non-Weierstrassian method of analytic prolongation.*

The present method arose from an attempt to find satisfactory necessary and sufficient conditions that a function of a real variable be an analytic function. It was found that this was so when and only when a certain straight line construction which makes a network of broken lines correspond to a subdivision of a certain portion of the plane into squares gives in the limit a network of curves through whose parameterization a transformation of the plane is defined whose corresponding functions have continuous total differentials. Since at the same time the method gives the values of the original function over a region of the complex plane, whereas originally it was known only along a segment of the real axis, it appears as a new method of analytic prolongation. By repeated application of the method, the function can be explored throughout its domain of existence. The method is that used by R. G. D. Richardson in 1917. In its particular development it yields a neat formula for the coordinates of an arbitrary point of intersection of the network as a finite series whose terms approach those of a Taylor's expansion.

43. Dr. E. L. Post: *A new method for generalizing e^x in the complex domain.*

The method in question is simply the one presented in the preceding paper. In the present case, the broken line network in the plane of the dependent variable becomes a set of lines radiating from the origin crossed orthogonally by a spider's web of broken lines. The lengths cut off along these radiating lines are found to be proportional to their distances from the origin, a fact which easily enables us to find that, in the limit, the network of broken lines is replaced by a pencil of lines crossed orthogonally by a series of concentric circles.

44. Dr. E. L. Post: *A simple geometric proof of the equality of the Brochardt angles of a triangle.*

It is easy to obtain a simple trigonometric proof of the equality of the Brochardt angles of a triangle, but as far as

the author is aware no simple geometric proof of this fact is extant. The present note supplies such a proof.

45. Professor I. J. Schwatt: *Method for the summation of series in which groups of positive terms alternate with groups of negative terms.*

The author considers certain series in which the groups of positive terms and the groups of negative terms alternate and contain m terms each. He represents these series as a sum of a finite number of integrals I_k, and he has obtained a formula which enables him to find the value of I_k and similar integrals without going through the cumbersome work of integration.

46. Mr. Hillel Poritsky: *On curves kinematically related to a given curve.*

This paper appeared in the February BULLETIN.

47. Professor F. L. Hitchcock: *A method for the numerical solution of integral equations.*

The purpose of this paper is to obtain a method requiring as little as possible in the way of evaluation of definite integrals. The Fredholm equation is treated over the interval from 0 to 1. It is shown that the only necessary definite integrals are (1) the integral of the given function $s(x)$ over this interval or over any desired finite part of this interval, and (2) the integral of the kernel $K(x, y)$ over a unit square or over any desired finite square contained within the unit square. The solution is obtained by a definite rule and the computation is suitable for carrying out with a calculating machine.

48. Professor Elizabeth B. Cowley: *Further generalizations of the old mathematical puzzle of three vessels with capacities of* 8, 5 *and* 3.

This is a further development along the lines of the paper presented at the October meeting. Here the problem of four vessels is studied, starting with the cases where $A = B + C + D$. It is found that, although certain cases can be solved by repeated use of one or both of the two methods employed for the three-vessel problem, it is necessary to devise other methods to meet the situation in cases where the three-vessel methods are not applicable. Some of these new methods can also be used in cases where the three-vessel methods are applicable. The conditions for the success or failure of the various methods are found. Certain modifications are necessary in cases where $A \geqq B + C + D$.

49. Mr. F. H. Murray: *Real solutions of certain linear differential equations of the second order.*

This paper is given to a study of the real bounded solutions of certain linear differential equations of a type discussed by Poincaré, Bohl, and Wiman. Besides a synthesis of some of the results obtained by these authors, certain new results are obtained by methods of successive approximations. In particular, an analytic expression for the general integral is obtained, from which many of the characteristic properties of the solutions follow immediately. A generalization of some of the results obtained by Bohl for linear equations with quasi-periodic coefficients is also given.

50. Mr. F. H. Murray: *A type of arc-describing instrument.*

In this paper is described a type of instrument* for drawing arcs of circles of very large radius.

51. Mr. F. H. Murray: *Certain isosceles triangle solutions in the problem of three bodies.*

The aim of this paper is to deduce a certain inequality, satisfied by the isosceles triangle solutions with axis of symmetry, and non-coplanar, in the problem of three bodies, for large negative values of the energy constant. From this inequality and another found by J. Chazy, one can show that the solutions under consideration possess "complete stability," in the sense of Poincaré (*Les Méthodes Nouvelles*, III).

52. Professor J. W. Alexander: *Invariant points of surface transformations.*

Let $A_1, A_2, \cdots, A_p, B_1, B_2, \cdots, B_p$ be a canonical system of cuts on a closed surface of genus p, and let Δ be a transformation of the surface into itself carrying the cuts A_i and B_i into the cuts $\overline{A}_i$ and $\overline{B}_i$ respectively. Then

$$\sum_j \tau_j = 1 + \tau + \sum_i \{N(B_i\overline{A}_i) + N(\overline{B}_iA_i)\},$$

where $\Sigma\tau_j$ denotes the sum of the indices of the fixed points of the transformation, τ the index of the transformation itself, and $N(AB)$ the intersection number of the curves A and B.

53. Mr. G. E. Raynor: *On Dirichlet's problem for irregular boundaries.*

The author gives an elementary solution of Dirichlet's problem under less restricted boundary conditions than those generally assumed.

R. G. D. RICHARDSON,
Secretary of the Society.

* These instruments are fully described in Letters Patent of the United States, No. 1,427,493, dated Aug. 29, 1922.

THE EVANSTON MEETING OF THE SOCIETY

The fiftieth regular meeting of the Chicago Section of the American Mathematical Society, being the eighteenth regular western meeting of the Society, was held at Northwestern University, Evanston, Illinois, on Friday, December 29, 1922. The attendance at this meeting was about forty-five, among whom were the following thirty-seven members of the Society:

Ballantine, W. S. Beckwith, A. D. Campbell, Chittenden, Coble, Curtiss, H. T. Davis, Dickson, Dowling, Dresden, Edington, Gokhale, Gouwens, Lane, Logsdon, McCain, McClenon, MacMillan, March, Marshall, T. E. Mason, C. N. Mills, E. J. Moulton, Plapp, Schottenfels, Skinner, J. H. Taylor, Theobald, E. L. Thompson, B. M. Turner, W. J. Wagner, R. L. Wilder, K. P. Williams, R. E. Wilson, F. E. Wood, Yanney, Zehring.

At the business meeting on Friday afternoon a resolution, proposed by Professor Skinner, expressing the Society's appreciation of the reception accorded them at Northwestern University was unanimously adopted. On Friday evening a dinner was held at the North Shore Hotel at which twenty-five persons were present. Toasts were responded to by Professors Holgate, Dickson, Curtiss, and Dresden.

At the sessions on Friday Professor Coble, Chairman of the Section, presided, relieved by Professor Curtiss in the forenoon and by Professor Dickson in the afternoon. Titles and abstracts of the papers read at this meeting are given below. Professor Moore's paper was presented by Mr. Wilder. The papers of Professor Jackson and Dr. Camp, and Professor Chittenden's first paper were read by title.

1. Professor E. P. Lane: *Ruled surfaces with generators in one-to-one correspondence.*

The author bases a projective theory of ruled surfaces, whose generators are in one-to-one correspondence and skew to each other, on a system of four ordinary linear first order differential equations in four dependent variables. This system is reduced to a canonical form by referring the surfaces to their intersector curves. The intersector curves, which are analogous to asymptotics, and certain other curves analogous to flecnode curves, are studied.

The general theory is applied in the investigation of Green-reciprocal ruled surfaces. Several of their properties are developed. In particular should be mentioned a characterization of the directrix congruences in terms of simple geometrical concepts.

2. Mr. R. L. Wilder: *Some theorems on continuous curves, with special reference to continuous curves that contain no simple closed curve.*

Among the theorems established are: (I) A sufficient* condition that a continuum M should be a continuous curve is that every two points that lie in a connected subset of an open subset of M lie in a sub-continuum of that open subset; (II) If N is a closed proper subset of a continuous curve M, then $M - N$ is a countable set of domains with respect to M, no two of which have a point in common, and if M contains no simple closed curve, then for $\epsilon > 0$, at most a finite number of these domains is of diameter $> \epsilon$; (III) A continuous curve M that contains no simple closed curve consists of (1) a sequence of arcs $C_1, C_2, C_3, \cdots$ every two of which have in common at most a point which is not interior to both, and such that (i) n being any positive integer, $C_1 + C_2 + \cdots + C_n$ is a continuous curve M_n and a proper subset of M, (ii) for $\epsilon > 0$ there exists a number ρ such that for $n > \rho$, $\delta(C_n) < \epsilon$ (δ meaning "diameter of") and $\delta(d) < \epsilon$, where d is any one of the maximal domains with respect to M lying in $M - M_n$; and (2) a totally disconnected set of non-cut points, P_ω, each of which is a limit point of the sequence $M_1, M_2, M_3, \cdots$, and containing all the non-cut points of M.

3. Mr. R. L. Wilder: *An analysis of the point set which constitutes the boundary of a complementary domain of a continuous curve.*

It is shown that the boundary β of a complementary domain of a continuous curve is the sum of, at most, a countable set of arcs and simple closed curves together with a totally disconnected set of limit points. It is also shown that: (I) Every closed and connected subset of β is itself a continuous curve, (II) β cannot contain, for any positive number ϵ, an infinite set of arcs of diameter $> \epsilon$, such that every two of

* R. L. Moore has proved this condition necessary. See this BULLETIN, vol. 28 (1922), p. 91.

these arcs have in common at most a point which is not an interior point of both, (III) If N is any connected subset of β, then every two points of N are the extremities of an arc lying wholly in N, (IV) A necessary and sufficient condition that the boundary of a simply connected domain be a continuous curve is that every connected subset of it should be connected in the strong sense.

4. Professor R. L. Moore: *An uncountable non-dense closed point set each of whose complementary intervals abuts on another one at each of its ends.*

This paper appeared in full in the February number of this BULLETIN.

5. Dr. Mayme I. Logsdon: *Closed sets of rational points on a plane cubic curve of genus one.*

If the elliptic argument of a rational point, A, on a plane cubic curve of genus one is commensurable with a period of the functions of the parameter which enter in the coordinate representation of the point, the process of using A to find other rational points by the method of tangentials and secants will terminate after a finite number of such points have been found. The author has studied certain configurations which may arise, in particular when the anharmonic ratio of the cubic is rational.

6. Mr. H. T. Davis: *Report on a boundary value problem of fourth order.*

The problem studied is that ot developing an existence theorem for the characteristic numbers λ_i of the system $L(u) + \lambda k(x)u = 0$, $U_i(u) = 0$ $(i = 1, \cdots, 4)$, where $L(u)$ is the most general self-adjoint expression of fourth order, the range of x is between two singular points and $U_i(u)$ are self-adjoint linear expressions in u, u', u'', u''', two of them being taken for $x = a$ and two for $x = b$. Real transformations of independent and dependent variables reduce the problem to a normal form, preserving the self-adjointness of the system. A criterion from the theory of integral equations is then applied and an existence theorem for the λ_i developed. The problem is shown to be the analogue of Sturm's boundary value problem for a system of second order. While the solution of the latter problem may be shown to depend upon the oscillation

of a certain solution of an equation of second order, the solution in the present case depends upon the oscillation of a certain solution of a fifth order equation due to Halphen.

7. Professor A. B. Coble: *The extension of the Weddle and Kummer surfaces to hyperelliptic 3-ways of genus 3.*

The author has previously shown that the generalized Kummer surface for $p = 3$, an M_3^{24} in S_7, can be defined as the double manifold of a certain quartic locus in S_7 whose equation has coefficients which are rational in the coordinates of an Aronhold set of 7 double tangents of the normal curve of genus 3—a ternary quartic. This definition fails completely for the hyperelliptic case since then the normal curve is a ternary quintic with a triple point. To cover this case the Kummer surface is obtained as a birational transform of a generalized Weddle surface—a 3-way in S_5—which is defined by 8 points in S_5 as the ordinary Weddle is defined by 6 points in S_3, i.e., as the locus of fixed points of a Cremona involution attached to the set of points.

8. Professor A. B. Coble: *Associated sets of points.*

A set of n points in S_k, P_n^k, is *associated* with a set Q_n^{n-k-2} if complementary minors of the matrices of the coordinates of the two sets are proportional. The author considers various methods of mapping the one set upon its associated set or of projecting the one upon the other. Properties of the set in the higher dimension are thus deduced from known properties of the associated set.

9. Professor L. E. Dickson: *The rational linear algebras of maximum and minimum ranks.*

The investigation relates to linear associate algebras with a principal unit, the coordinates of whose numbers range over an arbitrary field F. Such an algebra in n units of rank n is irreducible with respect to F if and only if $f(x)$ is a power of a polynomial irreducible in F, where $f(x) = 0$ is the rank equation. An algebra over F, of rank 2, has units $1, e_1, \cdots, e_m$ such that $e_i^2 = c_i$, $e_ie_j = -e_je_i$ $(i \neq j)$, where c_i belongs to F. If at least two c_i are not zero, then $m = 3$ and the algebra is the rational generalization of quaternions. If $c_i = 0$ $(i < m)$, $c_m \neq 0$, the algebra is equivalent either to

$$e_m^2 = 1, \qquad e_me_i = e_i, (i \leqq k), \qquad e_me_j = -e_j, (k < j < m),$$
$$e_re_s = 0, (r < m, s < m,)$$

where $2k \geqq m - 1$, or to

$e_m e_{2i-1} = e_{2i}$, $e_m e_{2i} = c_m e_{2i-1}$, $(i \leqq r)$, $e_j e_k = 0$, $(j < m, k < m)$, with $m = 2r + 1$, and c_m not a square in F. Finally, if every $c_i = 0$, the square of every number of the algebra is zero (a case of the outstanding problem of nilpotent algebras), and the rather numerous algebras in fewer than eight units are found. These results are applied to the determination of all algebras over any field F in 2, 3 or 4 units.

10. Professor L. E. Dickson: *A new simple theory of hypercomplex integers.*

The definition of a system of hypercomplex integers due to A. Hurwitz and applied to all classic algebras in 2, 3 and 4 units by Du Pasquier postulates rational coordinates, a finite arithmetical basis, closure under addition, subtraction and multiplication, the presence of the n basal units $e_0 = 1, \cdots, e_n$ (or only of e_0), and that the system is a maximal. Unfortunately, no maximal system exists for the majority of algebras. If we employ any non-maximal system, it usually happens that factorization into primes is not unique and cannot be made unique by the introduction of ideals of any kind. These essential difficulties all disappear if we replace the postulate of a finite basis by the assumption that, for every number of the system of integers, the coefficients of the rank equation are all rational integers.

11. Professor Arnold Dresden: *Symmetric forms in n variables.*

This paper considers the problem of expressing a symmetric form in n variables of weight S, of k parts and of given weight of each part, as a polynomial in the elementary symmetric functions of these variables. By means of a dominance relation between partitions of S, it is possible to specify which products of elementary symmetric functions will be involved. Various theorems are given for the determination of the coefficients with which these products will appear in the final result.

12. Professor Dunham Jackson: *A general class of problems in approximation.*

The results of recent papers by the author on the method of least mth powers are extended by the admission of a more

general function of the error in place of the mth power. If y denotes the error, $|y|^m$ is replaced by a function $E(y)$ which is defined and continuous for all real values of y, becomes positively infinite as y becomes infinite in either direction, and, in most of the theorems, possesses additional properties, varying from one theorem to another. Among the topics treated are the existence and uniqueness of the approximating function, the continuity of its dependence on the function for which an approximate representation is sought (a question which was not discussed explicitly in the earlier papers mentioned), and, in the case of approximation by finite trigonometric sums, the convergence of the approximating sum as its order is indefinitely increased. Some of the results are obtained by fairly direct application of the methods previously used; others call for new considerations of some delicacy.

13. Dr. W. E. Edington: *Abstract group definitions and applications.*

Three distinct infinite systems of non-abelian groups defined by the conditions $t_1^4 = t_2^4 = (t_1t_2)^2 = (t_1t_2^3)^\alpha = 1$, $t_1^3 = t_2^6 = (t_1t_2)^2 = (t_1^2t_2^2)^\alpha = 1$, and $t_1^3 = t_2^3 = (t_1t_2)^3 = (t_1t_2^2)^\alpha = 1$, are proved to exist for all values of α and the generators as substitutions are given. The groups are of order $4\alpha^2$, $6\alpha^2$, and $3\alpha^2$, respectively, and are all solvable. The first two systems are the groups of the elliptic norm curve C_α, respectively, when the invariants g_3 and g_2 are zero. The principal subgroups and the orders of the operators are determined for the groups in each system.

14. Dr. W. E. Edington: *On an infinite system of non-abelian groups of order nm^{n-1}.*

If two non-commutative operators s_1 and s_2 fulfil the conditions

$$s_1^n = s_2^n = (s_1^\alpha s_2^\alpha)^m = 1, \qquad s_1^\alpha s_2^\alpha s_1^\beta s_2^\beta = s_1^\beta s_2^\beta s_1^\alpha s_2^\alpha, \quad (\alpha, \beta = 0, 1, 2, \cdots, (n-1)),$$

they generate a group of order nm^{n-1}, and such a group exists for all values of m and n. The generators as substitutions are given in the paper. The groups of this system are all solvable. Several known infinite systems are special cases of this more general system.

15. Dr. W. E. Edington: *On an infinite system of non-abelian groups of order nm^n.*

Two non-commutative operators s_1 and s_2 having the

property that the transforms of their product s_1s_2 by s_1 and its powers are all distinct and commutative, and also fulfilling the conditions $s_1{}^n = s_2{}^{mn} = (s_1s_2)^m = 1$, generate a group of order nm^n, and groups exist for all values of m and n. The generators represented as substitutions also possess interesting properties.

16. Professor E. W. Chittenden: *Note on a property of abstract sets which admit a definition of distance.*

Fréchet gives the following definition of a complete set (D): a set (D) with preassigned elements of accumulation is *complete* if among all possible definitions of distance in (D) for which the elements of accumulation remain unchanged there is one with respect to which (D) admits a generalization of the theorem of Cauchy regarding the convergence of sequences; and remarks that it would be interesting to know whether there exist sets (D) which are not complete. It is shown that if a set (D) contains a set H which is in its derived set H', then (D) contains a perfect set. Therefore the rational numbers with distance defined as usual form a set (D) which is not complete.

17. Professor E. W. Chittenden: *The Schmidt linear differential forms of a limited bilinear form in infinitely many variables.*

The Hellinger theory of the linear differential forms of a quadratic form in infinitely many variables can be used to obtain a representation of the general limited bilinear form which is analogous to Schmidt's representation of the unsymmetric kernel $K(x, y)$ in terms of the fundamental functions of $\overline{K}(x, y) = \int_a^b K(x, t)K(y, t)dt$, $\underline{K}(x, y) = \int_a^b K(t, x)K(t, y)dt$.

18. Professor E. W. Chittenden: *On a form of the property of Borel-Lebesgue which is independent of the closure of derived classes.*

The closure of derived classes is extensionably attainable in terms of the concept, "limiting element of transfinite order." The property of Borel-Lebesgue and the property, "perfectly compact," of R. L. Moore, may be so generalized that the new properties are equivalent in any system (V) of Fréchet which has the property 2°, that any element of accumulation of the sum of two classes is an element of accumulation of at least

one of the two classes. Furthermore, if derived classes are closed (in the usual sense), the generalized properties are equivalent, respectively, to the property of Borel-Lebesgue and the property, "perfectly compact."

19. Professor K. P. Williams: *Concerning an expansion in the restricted problem of three bodies.*

In one of the Lagrange particular solutions of the problem of three bodies, the function $y(\mu)$ defined by

$$y^5 + (3 - \mu)y^4 + (3 - 2\mu)y^3 - \mu y^2 - 2\mu y - \mu = 0$$

comes into consideration (Moulton, *Celestial Mechanics*, 1st edition, p. 197). The purpose of the present paper is to determine the radius of convergence of the series that gives y.

20. Dr. C. C. Camp: *Expansions in terms of solutions of partial differential equations.* Second paper.

The author considers the equation $\sum_{i=1}^{p} L_i(u) + \lambda u = 0$ with the boundary conditions $T_{ij}(u) = 0$, $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, n_i$, where $L_k(u)$ is a partial differential expression of order n_k in x_k similar to the ordinary differential expression in x considered by Birkhoff in his thesis, and $T_{kj}(u)$ is a partial differential expression of order $n_k - 1$ such that $T_{kj}(u) = 0$ reduces to one of Birkhoff's *regular* boundary conditions in a_k and b_k by the transformation $u = p\prod_{i=1}^{p} u_i(x_i)$. This gives rise to a set of p Birkhoff systems, each involving a different parameter. By employing a convergence proof similar to that used in his first paper the author shows that a function f in p variables, similarly restricted in the region $a_i \leqq x_i \leqq b_i$, may be expanded analogously in a *multiple Birkhoff series* in terms of principal solutions of the set of systems and of the adjoint set. Such a series has the same convergence properties as the multiple Fourier expansion except at points on the boundaries. The development has applications in important physical problems such as those of potential, heat flow, and wave motion.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

THE FIFTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The fifteenth regular meeting of the Southwestern Section of this Society was held at the University of Kansas on Saturday, December 2, 1922. The total attendance was twenty-eight, including the following sixteen members:

Ashton, Black, Candy, Garrett, Ingold, Kendall, Lefschetz, J. V. McKelvey, U. G. Mitchell, R. L. Moore, Roever, Edwin R. Smith, G. W. Smith, Stouffer, J. S. Turner, Wheeler.

Professor Ashton occupied the chair, being relieved during the afternoon session by Professor Lefschetz. The morning session and part of the afternoon session were devoted to the reading of the papers listed below. During the afternoon session Professor R. L. Moore gave a special lecture by invitation on *Continuous curves from the view-point of analysis situs*. This paper will appear in full in a later number of this BULLETIN. It was voted to hold the next meeting of the Southwestern Section at the University of Missouri. The following program committee was elected: Professors Hedrick (chairman), McKelvey, and Stouffer (secretary).

The titles and abstracts of the papers read follow below.

1. Professor A. L. Candy: *Cyclic operations on determinants.*

This paper consists chiefly of three theorems: (I) If the rows of a given determinant Δ be added cyclically s in a set, the resulting determinant Δ_s will be equal to zero, or $s \cdot \Delta$, according as n (order of Δ) and s are commensurable, or incommensurable. (II) If the elements in each column of a determinant Δ be permuted cyclically in all possible ways, and the n^n determinants thus formed be collected into groups such that each determinant in the group shall have the same subscripts in the first row, the sum of the determinants in any one of these groups will be equal to $\pm \lambda\Delta$, or zero, according as the weight of the group (the sum of the subscripts in any first row) is, or is not, a multiple of n, where λ is 1, n, a submultiple of n, or the sum of two or more such numbers. (III) The sum of the $n!$ determinants that can be formed by making the $n!$ permutations of the columns of a circulant is equal to zero.

2. Professors Claribel Kendall and G. W. Smith: *On the conditions for associativity in linear algebra.*

In a linear algebra of n units the differences $(e_ie_j)e_k - e_i(e_je_k)$ lead to n^4 quadratic forms in the n^3 constants of multiplication. The authors establish certain relations among these forms, and in the case $n = 2$ determine eight relations among them.

3. Professor G. W. Smith: *Analytic expression for line segments.*

In this paper the author sets up an analytic expression which represents $f(x)$ in a certain interval $a < x < b$ $(a > 0)$ and only in this interval, the expression being imaginary for other values of x. By a combination of these he represents analytically such geometric figures as triangles and squares.

4. Professor W. H. Roever: *Singularity in the boundary curve of the projection of a surface.*

In the boundary curve of the projection of a surface there may be points at which the curve stops abruptly. This is illustrated in the case of the orthographic projection of a torus, for which the curve in question consists of two parallels to an ellipse. If the torus is sufficiently inclined to the plane of projection the inner parallel has four cusps, and only that portion of it which lies between a particular pair of these cusps is visible and thus the inner boundary apparently stops at these points. Several methods of determining such singularities.

5. Professor Edwin R. Smith: *The determination of hidden periodicities by difference methods.*

If $y = f(x, \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n)$ is a function of x with n distinct simple unknown periods $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$, it is shown that the function satisfies a difference equation of the form $u(x) - C_1u_{n-1}(x) - C_2u_{n-2}(x) - \cdots - (-1)^nC_nu_0(x) = 0$, where $C_1, C_2, C_3, \cdots, C_n$ are symmetric functions of $\cos \alpha_1$, $\cos \alpha_2, \cdots, \cos \alpha_n$ and $u_0(x) = y_{x+1} - y_x$, $u_i(x) = u_{i-1}(x - 1) - u_{i-1}(x + 1)$, $(i = 1, 2, \cdots, n)$. The values of $C_1, C_2, C_3, \cdots, C_n$ corresponding to a given set of observations may be found by any other convenient method. The method of least squares leads to results similar to those obtained by the Lagrange-Dale method.

6. Professor S. Lefschetz: *Concerning Nöther's theorem on curves traced on non-singular surfaces.*

The theorem in question states that *all algebraic curves traced on a generic surface of order $m > 3$ in ordinary space are complete intersections.*

In his Bordin prize paper the author gave the first complete proof of this important proposition, or rather of a more general one which includes it as a special case. One of the main features of the proof consists in showing that *not* all double integrals of the first kind attached to the surfaces in question have all periods equal to zero. In this paper a notable simplification of this part of the proof is given, applicable when it is desired to prove strictly Nöther's theorem and no more. It consists essentially in building up a surface of order m which has an almost obvious integral with a non-zero period.

7. Professors E. R. Hedrick and Louis Ingold: *Remarks concerning the Michelson-Morley experiment.*

Because of prevailing differences of opinion relating to the celebrated Michelson-Morley experiment a new investigation of the mathematical theory seems to be desirable. The work confirms, by an independent calculation, certain results obtained by Righi, and this is of great importance since the accuracy of his work has been called in question (OBSERVATORY, vol. 44 (1921), pp. 340–341).

8. Professor J. S. Turner: *Extensions of the Euler and Seelhoff determinants, with examples of their use in determining whether a number is prime.*

It is first pointed out that the non-simple reduced forms of the odd Euler determinants become simple forms, i.e., forms containing no product term when multiplied by 2. Hence if a prime p has the quadratic character of a non-simple reduced form, then $2p$ has a single representation by a simple form. A list of negative determinants follows in which there are three reduced forms in each genus, such that four times each non-simple form is a simple form. If p has the quadratic character of a simple form, $4p$ has a single representation by that form. Seelhoff's list (AMERICAN JOURNAL, vols. 7–8) of determinants in which certain genera contain only simple reduced forms is extended. A list of determinants follows in which certain genera contain only ambiguous reduced forms, i.e., forms which are simple or become simple on multiplication by 2. The final list contains determinants for which all the forms in certain genera are simple or become simple on multiplication by 4. The advantages of the new determinants are explained and illustrated by examples. These examples also illustrate an improved method of exclusion.

E. B. STOUFFER, *Secretary of the Section.*

A SET OF AXIOMS FOR LINE GEOMETRY*

BY M. G. GABA

1. *Introduction.* In 1901 Pieri proposed a set of axioms for line geometry in terms of line and intersection.† That Pieri's set of eleven postulates was not independent was shown by Hedrick and Ingold in 1914; they proposed a simpler and more elegant set of but five independent axioms, using the same undefined concepts.‡ Both of these sets are for geometries equivalent to the general three-space geometry established by axioms A_1, A_2, A_3, E_0, E_1, E_2, E_3 and E_3' of Veblen and Young.§

In this paper is given a set of six independent axioms in terms of line as an undefined element and an undefined class of one-to-one correspondences among the lines called collineations. There is introduced but one defined term before the complete statement of the axioms. To make a proper projective space it has usually been necessary not only to add a postulate of projectivity but also a sequence of definitions for such concepts as perspectivity, projectivity, etc., to give that postulate content. If to our set a seventh postulate is added, we have a proper projective three-space without the intervention of any additional defined concepts.‖

2. *Postulates.* Our basis is a class of undefined elements, called *lines*; an undefined class of one-to-one correspondences, or transformations, among the lines, called *collineations*; and

* Presented to the Society, Nov. 27, 1920.

† *Sui principi che regno la geometria delle rette*, TORINO ATTI, vol. 36 (1901), pp. 335–351.

‡ *A set of axioms for line geometry*, TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 205–214.

§ *A set of assumptions for projective geometry*, AMERICAN JOURNAL, vol. 30 (1908); *Projective Geometry*, vol. 1, Boston, 1910.

‖ Another set of postulates equivalent to the set of all seven of the axioms is the first seven given by the author in his paper *A set of postulates for general projective geometry*, TRANSACTIONS OF THIS SOCIETY, vol. 16 (1915), pp. 51–61.

a class of defined classes of lines, called *fields*. By a one-to-one correspondence among the lines we mean that to every line α there corresponds one and only one line α', and no α' is the correspondent of two distinct lines α_1 and α_2. This concept of one-to-one correspondence may be said to belong to the domain of pure logic; we shall therefore not consider it in our basis as an undefined mathematical notion.* We shall define a field as a class of at least three lines such that: (1) every collineation that makes any two distinct lines of the class correspond to two distinct lines of the class leaves the class invariant; (2) a collineation exists that will make any two distinct lines of the class correspond to any two distinct lines of the class; (3) the class has no proper super-class possessing property (2).

Our axioms, whose technical independence is shown by examples given at the end of this paper, are as follows:

I. *There exists a line.*

II. *If α is a line, there exist five distinct fields containing α.*

III. *If a collineation τ_1 exists such that τ_1 makes correspond to the set of lines A the set of lines B, and if a collineation τ_2 exists such that τ_2 makes correspond to the set of lines A the set of lines C, then a collineation τ_3 exists such that τ_3 makes correspond to the set of lines B the set of lines C.*

IV. *If no collineation exists that makes correspond to the field A the field B, and if no collineation exists that makes correspond to the field A the field C, then a collineation τ exists that makes correspond to the field B the field C.*

V. *If A and B are fields such that a collineation τ exists that makes correspond to the field A the field B, then the fields A and B have a line in common.*

VI. *If A, B and C are fields such that A and B have a line α_1 in common, and B and C have a line α_2 in common, and C and*

* It is true, of course, that no sharp line of demarcation exists between concepts of pure logic and of mathematics; but an author may choose whether or not he will include a given concept in his own discussion. See Federico Enriques, *Problems of Science*, authorized translation by J. Royce. Open Court Publishing Co., 1914, p. 122.

A have a line α_3 in common, then there exists a field D such that D contains the lines α_1, α_2 and α_3.

VII. *If A, B, C and D are four distinct fields having a line α in common, and if collineations τ_1, τ_2 and τ_3 exist such that τ_1 makes correspond to the field A the field B, τ_2 makes correspond to the field B the field C, and τ_3 makes correspond to the field C the field D, then every collineation τ that leaves A, B and C invariant also leaves D invariant.*

3. *Theorems.* THEOREM 1. *If A is a field, then a collineation exists that leaves A invariant.*

Let us assume that no collineation exists that leaves A invariant. If now we let fields B and C of Postulate IV be identical with A, we are led at once to an absurdity.

THEOREM 2. *If a collineation τ exists that makes correspond to the field A the set of lines B (or transforms A into B, in notation, $\tau(A) = B$), then a collineation τ^{-1} exists such that τ^{-1} makes correspond to the set of lines B the field A: $\tau^{-1}(B) = A$.*

By hypothesis, a collineation τ exists such that $\tau(A) = B$, and, by Theorem 1, a collineation τ_1 exists such that $\tau_1(A) = A$. Hence it follows from Postulate III that a collineation τ^{-1} exists such that $\tau^{-1}(B) = A$.

THEOREM 3. *If a collineation τ_1 exists such that τ_1 transforms the field A into a set of lines B, and a collineation τ_2 exists such that τ_2 transforms the set of lines B into a set of lines C, then a τ_3 exists such that τ_3 transforms A into C.*

By Theorem 2, a collineation ${\tau_1}^{-1}$ exists such that ${\tau_1}^{-1}(B) = A$. The existence of ${\tau_1}^{-1}$ and τ_2, by virtue of Postulate III, implies the existence of a collineation τ_3 such that $\tau_3(A) = C$.

COROLLARY. *The resultant of a sequence of collineations on a field is a collineation.*

THEOREM 4. *Every collineation transforms a field into a field.*

Let τ be any collineation that transforms the field A into a set of lines B. Let β_1, β_2, β_1' and β_2' be any four lines of B such that β_1 and β_2 are distinct and β_1' and β_2' are distinct; and let α_1, α_2, α_1' and α_2' be the four lines of A such that $\tau(\alpha_1, \alpha_2, \alpha_1', \alpha_2') = \beta_1, \beta_2, \beta_1', \beta_2'$. Let τ_1 be any collineation such that $\tau_1(\beta_1, \beta_2) = \beta_1', \beta_2'$. From Theorem 2 and the coroll-

ary of Theorem 3, we know that a collineation $\tau_2 = \tau^{-1}\tau_1\tau$ exists such that $\tau_2(\alpha_1, \alpha_2) = \bar{\alpha}_1', \bar{\alpha}_2'$, where $\bar{\alpha}_1'$ and $\bar{\alpha}_2'$ are lines of A. But since the collineation τ_2 makes two lines of the field A correspond to two lines of A (which are distinct, since they are the correspondents of pairs of distinct lines), τ_2 leaves the field A invariant. Therefore, since $\tau^{-1}\tau_1\tau(A) = A$, $\tau(A) = B$, and $\tau^{-1}(B) = A$, we have $\tau^{-1}\tau_1(B) = A$, and hence $\tau_1(B) = B$.

Since A is a field, a collineation τ_3 exists such that $\tau_3(\bar{\alpha}_1, \bar{\alpha}_2) = \alpha_1', \alpha_2'$, where $\tau^{-1}(\beta_1, \beta_2) = \bar{\alpha}_1, \bar{\alpha}_2$. Therefore a collineation $\tau_4 = \tau\tau_3\tau^{-1}$ exists such that $\tau_4(\beta_1, \beta_2) = \beta_1', \beta_2'$.

We have shown that properties (1) and (2) of a field are possessed by the set of lines B. Let us now assume that B has a proper super-class B^* also possessing property (2). If $\tau'(B) = A$ and if $\tau'(B^*) = A^*$, then A^* is a proper super-class of A. Let α_1^*, α_2^* and $\alpha_1^{*\prime}, \alpha_2^{*\prime}$ be any two pairs of distinct lines of A^*. Furthermore let $\beta_1^*, \beta_2^*, \beta_1^{*\prime}$ and $\beta_2^{*\prime}$ be those four lines of B^* such that $\tau(\alpha_1^*, \alpha_2^*) = \beta_1^*, \beta_2^*$ and $\tau^{-1}(\beta_1^{*\prime}, \beta_2^{*\prime}) = \alpha_1^{*\prime}, \alpha_2^{*\prime}$. By our assumption, a τ_5 exists such that $\tau_5(\beta_1^*, \beta_2^*) = \beta_1^{*\prime}, \beta_2^{*\prime}$; therefore a $\tau_6 = \tau^{-1}\tau_5\tau$ exists. But $\tau_6(\alpha_1^*, \alpha_2^*) = \alpha_1^{*\prime}, \alpha_2^{*\prime}$; hence A^* possesses property (2) of a field, which is impossible if A is a field. Therefore our assumption leads to a contradiction; hence B is a field.

THEOREM 5. *If A and B are distinct fields such that a collineation exists transforming A into B, then A and B have at most one line in common.*

Let us assume that A and B have the two distinct lines γ_1 and γ_2 in common. If the collineation that makes A go over into B is τ, then let $\tau(\gamma_1, \gamma_2) = \beta_1, \beta_2$. Since $\gamma_1, \gamma_2, \beta_1$ and β_2 are pairs of distinct lines of B, a collineation τ_1 exists such that $\tau_1(\beta_1, \beta_2) = \gamma_1, \gamma_2$ and $\tau_1(B) = B$. But $\tau_1\tau(A) = A$, since γ_1 and γ_2 of A are invariant; and $\tau_1\tau(A) = B$, since $\tau(A) = B$; therefore our assumption must be wrong.

THEOREM 6. *If A is a field containing the lines α_1 and α_2, then there exists a field $\bar{A}$ also containing α_1 and α_2 and such that no collineation exists that will transform A into $\bar{A}$.*

We shall assume α_1 and α_2 distinct, since the theorem is true in the case where α_1 and α_2 are identical if it is true when α_1

and α_2 are distinct. By means of Postulate II we know that there exists a field A_1 containing the line α_1 and distinct from A. Let τ be a collineation such that $\tau(\alpha_1, \alpha_2) = \alpha_2, \alpha_1$. From Theorem 4, we have $\tau(A_1) = A_2$, where A_2 is a field containing α_2. If $A_1 = A_2$, then A_1 is $\overline{A}$, since A and A_1 have the two distinct lines α_1 and α_2 in common; hence they cannot be transformable into one another without contradicting Theorem 5. Let us now assume A_1 and A_2 distinct. Since A_1 is transformable into A_2, fields A_1 and A_2 have a line α in common. But since A and A_1 have the line α_1 in common, A_1 and A_2 the line α in common, and A_2 and A the line α_2 in common, it follows from Postulate VI that a field A' exists containing α, α_1 and α_2. If A is not transformable into A', then A' is $\overline{A}$. If A is transformable into A', then A is identical with A' by Theorem 5. In the latter case, however, A_1 is not transformable into A, since if that were possible, A and A_1 would be identical, since they have the lines α_1 and α (distinct, since otherwise $A_1 = A_2$) in common. Since α, α_1 and α_2 are distinct lines of the field A, a collineation τ_1 exists such that $\tau_1(\alpha_1, \alpha) = \alpha_1, \alpha_2$. The collineation τ_1 transforms the field A_1 into a field A_1' containing the lines α_1 and α_2. But A cannot be transformed into A_1', because, if that were possible, then, by Postulate III, A_1 would be transformable into A. Hence it follows that A_1' is $\overline{A}$ in this case.

THEOREM 7. *Every field belongs to one or to the other of two classes of fields such that collineations exist that will transform any field of one class into any field of the same class, and such that no collineations exist that will transform any field of one class into a field of the other class.*

From Theorem 6, we know that two fields A and $\overline{A}$ exist such that no collineation will transform A into $\overline{A}$. Let $[A_i]$ be the class of all fields transformable into A, and let $[\overline{A}_i]$ be the class of all fields transformable into $\overline{A}$. Let B be any field. If a collineation τ exists such that $\tau(B) = A$, then B belongs to $[A_i]$. Suppose B does not belong to $[A_i]$, then since no collineation exists transforming B into A, and no collineation exists transforming $\overline{A}$ into A, it follows from Theorem 2 and

Postulate IV that a collineation $\bar{\tau}$ exists such that $\bar{\tau}(B) = \overline{A}$; hence B belongs to $[\overline{A}_i]$. But B cannot belong to both $[A_i]$ and $[\overline{A}_i]$, for in that case it would follow from Postulate III that A could be transformed into $\overline{A}$. Any field A_i of the class $[A_i]$ will determine the class, since it is easy to show that if A_i and A_j are both members of the class and hence transformable into A, they are transformable into one another. Furthermore, no A_i of $[A_i]$ can be transformed into an $\overline{A}_j$ of $[\overline{A}_i]$, since that would make A_i belong to both classes.

We shall call the fields of one of the two classes of fields *points*, and the fields of the other class *planes*. We shall use the terminology α is a line *on* the plane p or p is *on* α to mean α is a line of p; P is a point *on* the plane p or p *on* P to mean that P and p have a line in common; P is a point *on* the line α or α *on* P to mean that α is a line of P. If two lines are on same point, they are called *copunctal;* if two lines are on the same plane, they are called *coplanar.*

THEOREM 8. *If a point P is on the line α and α is on the plane p, then P is on p.*

Since P and p have the line α in common, the theorem is obvious.

THEOREM 9. *If two lines are copunctal (coplanar), they are coplanar (copunctal).*

This theorem is but a restatement of Theorem 6 in the light of subsequent definitions.

THEOREM 10. *If P_1 and P_2 are distinct points (planes), then there is at least one line on both P_1 and P_2.*

See Postulate V.

THEOREM 11. *If P_1 and P_2 are distinct points (planes), there is not more than one line on both P_1 and P_2.*

See Theorem 5.

Points (planes) which are on the same line are said to be *collinear;* points (planes) which are not on the same line are said to be *non-collinear.*

THEOREM 12. *If P_1, P_2 and P_3 are three non-collinear points (planes), then there is one and only one plane (point) p such that P_1, P_2 and P_3 are on p.*

Let the line common to P_1 and P_2 be α_3, the line common to P_2 and P_3 be α_1, and the line common to P_3 and P_1 be α_2. By Postulate VI a field exists containing α_1, α_2 and α_3. If this field is a point (plane), then P_1, P_2 and P_3 coincide, contrary to the hypothesis that they are non-collinear and hence distinct. The field is therefore a plane (point). There can be but one such plane (point). Since α_1 and α_2 are distinct (otherwise P_1, P_2 and P_3 are collinear), they determine the plane (point) p. Let α_1' be a line of P_1 in p', α_2' a line of P_2 in p', and α_3' a line of P_3 in p'; hence P_1, P_2 and P_3 are on the plane (point) p'. If α_1' and α_2' coincide, then $\alpha_1' = \alpha_2' = \alpha_3$. Since α_1' and α_2' are coplanar (copunctal), they must be copunctal (coplanar). If α_1' and α_2' are distinct, they determine a point (plane) P_3'. Now P_3' cannot be on α_3 since that would make α_1' and α_2' coincide. By the part of the theorem already proved, a plane (point) p'' exists such that P_1, P_2 and P_3' are on p''. But p'' contains α_1' and α_2'; therefore p'' is p'. Hence the plane (point) p' contains α_3. In like fashion it can be shown that p' contains α_1 and α_2; hence p' is p.

COROLLARY 1. *If P_1 and P_2 are distinct points (planes), and if P_1 and P_2 are both on the plane (point) p, then the line α common to P_1 and P_2 is on P.*

COROLLARY 2. *If P is a point (plane) and α is a line such that P is not on α, then there is one and only one plane (point) determined by P and α.*

THEOREM 13. *If P_1, P_2, P_3 are three points (planes) not on the same line, and α is a line joining P_2 and P_3, the class $[P_i]$ of all points (planes) such that every point (plane) of $[P_i]$ is collinear with P_1 and some point (plane) on α, is such that every point (plane) of $[P_i]$ is on the plane (point) p determined by P and α, and every point (plane) on the plane (point) p is a point (plane) of $[P_i]$.*

Let P_i be a point (plane) collinear with P_1, and P_i a point (plane) on α. P_i is on p, since P_i is on α, by Theorem 8. Let the line determined by P_1 and P_i be α_i. Then α_i is on p by Corollary 1, Theorem 12. But P_i is on α_i; therefore P_i is on p.

Let P' be any point (plane) of p. P_1 and P' determine a line α' (unless P' is P_1, in which case P' obviously belongs to $[P_i]$). The lines α' and α are distinct since α is not on P_1, and they are copunctal (coplanar) since both lines are on p. Hence α and α' determine a point (plane) $\overline{P'}$. Hence P' is collinear with P_1, and $\overline{P'}$ a point (plane) on α; consequently P' belongs to $[P_i]$.

THEOREM 14. *If P_1, P_2, P_3 are points (planes) not all on the same line, and P_4 and P_5 ($P_4 \neq P_5$) are points (planes) such that P_2, P_3, P_4 are on a line and P_1, P_3, P_5 are on a line, there is a point (plane) P_6 such that P_1, P_2, P_6 are on a line and also P_4, P_5, P_6 are on a line.*

Since the points (planes) P_1, P_2 and P_3 are non-collinear, they determine a plane (point) p. The three lines α_1 common to P_2 and P_3, α_2 common to P_1 and P_3, and α_3 common to P_1 and P_2, are on p by Corollary 1, Theorem 12. Since P_4 is on α_1, and P_5 is on α_2, P_4 and P_5 are also on p. Hence the line α common to P_4 and P_5 is on p, and consequently α is copunctal (coplanar) with α_1. If α and α_1 are distinct, they determine a point (plane) which is P_6. If α and α_1 coincide, then any point (plane) on α will serve as a P_6.

THEOREM 15. *If α_1 and α_2 are lines, then α_1 and α_2 are on the same number of points (planes).*

Since collineations exist that transform any point (plane) into any point (plane) and any line in a point (plane) into any other in the same point (plane), a collineation τ exists that transforms α_1 into α_2. Every point (plane) on α_1 is transformed by τ into a point (plane) on α_2, and every point (plane) on α_2 comes from a point (plane) on α_1. Hence α_1 and α_2 are on the same number of points (planes).

THEOREM 16. *All points (planes) are not on the same line.*

Let us assume that every point (plane) is on a line α. Let τ be a collineation that transforms α into a distinct line α'. If we transform every point (plane) by τ, we get all the points (planes) again, since distinct fields transform into distinct fields. But all points (planes) now contain α' as well as α. Therefore it follows that there is but one point (plane) P.

Every line is in P, since, by Theorem 6, every line is in a point (plane). No other field can exist, since, if it did, it would be a proper sub-class of P, contrary to the definition of field. But if there is only one field, Postulate II is contradicted. Hence our original assumption is false and the theorem proved.

THEOREM 17. *If α is a line, then the number of planes on α is the same as the number of points on α.*

Since, by Theorem 16, all planes are not on α, there exists a line $\bar{\alpha}$ non-coplanar with α. Every point on $\bar{\alpha}$ determines, with the line α, a different plane, since if two distinct points $\bar{P}_1$ and $\bar{P}_2$ on $\bar{\alpha}$ and α were on the same plane, the lines $\bar{\alpha}$ and α would be coplanar. Hence there are at least as many planes on α as there are points on $\bar{\alpha}$. Likewise every plane on α determines, with the line $\bar{\alpha}$, a different point. Hence there are at least as many points on $\bar{\alpha}$ as there are planes on $\bar{\alpha}$. Since, by Theorem 15, the number of points on α and on $\bar{\alpha}$ are equal, and since the number of planes on α must equal the number of points on $\bar{\alpha}$, it follows that α is on the same number of points as α is on planes.

THEOREM 18. *There are at least three points (planes) on every line.*

By Postulate II, every line is on at least five fields; and, since the number of planes and points on a line is the same, the theorem follows immediately.

THEOREM 19. *All points (planes) are not on the same plane (point).*

Let p be any plane (point). Since there is more than one plane (point), there exists a line α not on p. If two points (planes) of α were on p, then α would be on p. Hence at most one of the points (planes) of α can be on p.

4. *Comparison with Ordinary Geometry.* Let us now compare our geometry with the general three-space geometry of Veblen and Young.* If we identify our lines as lines, collineations as projectivities, and fields as either points or planes in the Veblen-Young geometry, then our axioms are theorems in

* Loc. cit.

their geometry. Incidently, the consistency of our axioms is thus established.

On the other hand, let the undefined *line* of Veblen and Young be our *line*; their undefined *point*, our *point*; and their *on* our *on*. Consider their assumptions of alignment A:

A_1. *If A and B are distinct points, there is at least one line on both A and B.*

A_2. *If A and B are distinct points, there is not more than one line on both A and B.*

A_3. *If A, B, C are points not all on the same line, and D and E ($D \neq E$) are points such that B, C, D are on a line and C, A, E are on a line, there is a point F such that A, B, F are on a line and also D, E, F are on a line.*

We see that, except for notation, A_1 is our Theorem 10, A_2 is Theorem 11, and A_3 is Theorem 14. The *plane* defined by Veblen and Young is equivalent to our *plane*, by virtue of Theorem 13.

The assumptions of extension, substituting for the usual E_3' an equivalent axiom also given by Veblen and Young, are as follows:

E_0. *There are at least three points on every line.*

E_1. *There exists at least one line.*

E_2. *All points are not on the same line.*

E_3. *All points are not on the same plane.*

E_3'. *Any two distinct planes have a line in common.*

E_0 is our Theorem 18; E_1 is Postulate I; E_2 is Theorem 16; E_3 is Theorem 19; and E_3' is Theorem 10. We have therefore proved that our geometry is equivalent to that of Veblen and Young.

The assumption of projectivity is as follows:

P. *If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant.*

It is evident that P is a rephrasing of Postulate VII in terms of point.

5. *Independence Examples.*—1. No lines exist. Hence neither collineations nor fields exist.

—2. Lines are all the lines of a projective three-space and

an extra line not in that three-space. Collineations are all the projectivities on the lines of the three-space, but in addition leave the extra line invariant. Fields are therefore the points and planes of the three-space.

—3. Lines are all the lines of a projective finite three-space. Collineations are all the projectivities on the three-space and an extra transformation τ, such that τ transforms every line α into a line α' skew to α. Fields are therefore the points and planes of the three-space.

—4. Lines are all the lines of two projective three-spaces having no line in common. Collineations are all the projectivities on the two three-spaces such that each of the three-spaces is left invariant. Fields are therefore the points and planes of the two three-spaces.

—5. Lines are all the lines of two projective three-spaces having no line in common. Collineations are all the projectivities on the two three-spaces such that each of the three-spaces is left invariant, or such that the two three-spaces are interchanged. Fields are therefore the points and planes of the two three-spaces.

—6. Lines are all the lines of a euclidean plane. Collineations are all the projectivities such that parallel lines are transformed into parallel lines. Fields are therefore pencils of lines.

—7. Lines are all the lines of an improperly projective three-space. Collineations are all the projectivities in such a three-space. Fields are therefore the points and planes of that three-space.

UNIVERSITY OF NEBRASKA

SHORTER NOTICES

Théorie Mathématique des Assurances. By P. J. Richard et E. Petit. Second edition, revised throughout and brought up to date. Paris, Librairie Octave Doin, Gaston Doin, Editeur, 1922. Volume I, 455 pp.; Volume II, 320 pp.

These two volumes appear in the series entitled ENCYCLOPÉDIE SCIENTIFIQUE–BIBLIOTHÈQUE DE MATHÉMATIQUES APPLIQUÉES. The first volume, book one, contains two chapters on the theory of probability, with development of elementary theorems, the theorem of Bernoulli, Baye's rule, and a consideration of insurance from the point of view of a game of chance.

Book two contains a chapter on life and mortality tables and gives definitions of the rate of mortality and derived functions. The effects of age, sex, occupation, climate, selection, and other causes on the rate of mortality are explained. Various methods of constructing and graduating the rate of mortality are described, including graphical, mechanical and analytic adjustment.

Several proposed laws of human mortality including the formulas of De Moivre, Gompertz and Makeham and other generalizations are given. The old and the modern French tables for annuitants and for insured are described together with methods of adjustment and graduation to Makeham's law.

Chapter II is divided into four parts. The first part relates to annuities-certain and life annuities and develops the general theory of various types of annuities and their expression in terms of commutation symbols. Part two deals with last survivor annuities and annuities to x after the death of y. The theory of single premium insurance benefits on one life and on joint lives is developed in part three and the last part deals with a variety of forms of insurance and gives methods of deriving the net single and annual premiums.

Chapter III consists of five parts devoted respectively to gross premiums and methods of loading, mathematical reserves, contracts of insurance and alterations, dividends in insurance, and concludes with a study of the mathematical insurance risk and the stability of insurance based on the theory of risk.

This volume concludes with a very good six page bibliography of life insurance literature.

The second volume is divided into two books. The first book is devoted to insurance on the individual and consists of five chapters.

The first chapter develops very fully the fundamental probabilities in the theory of insurance against disability, the construction of disability tables, of rates of mortality among the disabled, and so on. A brief account is given of published statistics and of disability tables in use. It is curious that no mention is made of Hunter's *Disability Table*, the one most used in this country.

The second chapter contains an explanation of formulas for disability premiums and reserves under various conditions.

In chapter III there is given an account of the theory and derivation of rates of sickness, of the statistics pertaining thereto, the principal sickness tables, and the methods of calculating net and gross sickness premiums and the mathematical reserves.

Chapters IV and V have to do with applications of the combination of insurance against sickness, disability and death, and also a variety of other forms of insurance relating to marriage, birth, civil responsibility, and so on.

Book II, which deals with collective insurance, gives the technical basis of social insurance and special treatment of insurance against travel accident. An appendix contains an account of the operation of the Caisse Nationale for insurance against death, accident, and to provide for old age. It contains also a brief history and treatment of tontines and societies operating on a tontine basis, and an account with formulas of certain financial operations on a collective basis.

The second volume closes with seven pages of bibliographic references of much value to students of the subjects mentioned therein.

In the opinion of the reviewer these two books cover an unusual variety of insurances, especially the second volume, and the clear and complete development of theory must meet with general approval among mathematicians interested in these subjects. The practical applications are taken chiefly from foreign sources and have little bearing on the methods used in this country.

J. W. GLOVER

Leçons sur les Invariants Intégraux. By E. Cartan. Paris, Hermann, 1922. x + 210 pp.

The theory of integral invariants was first set forth by H. Poincaré in the third volume of *Les Méthodes Nouvelles de la Mécanique Céleste.* Cartan develops this theory systematically in his Leçons, using a different point of view. Instead of connecting an integral invariant with a system of differential equations as did Poincaré, Cartan considers the integrand of this integral as a differential form and studies its properties of invariance under a group of transformations.

The mathematician will be especially interested to read this book because of the elegant treatment of the subject. Enough of the classical theories of differential forms and continuous groups, for instance, is given to make the book readable. Considerable space is devoted to the development of methods of deriving integral invariants, such as Jacobi's last multiplier and infinitesimal transformations.

These Leçons are to be recommended to students of applied mathematics because of the admirable treatment of physical problems from the modern point of view. The theory of turbulence, the n-body problem, and certain problems in optics are considered from the point of view of tensors and their relation to the integral invariant theory.

E. P. LANE
H. T. DAVIS

NOTES

The Summer Meeting of the Society will be held at Vassar College, September 6–8, 1923, in conjunction with the meeting of the Mathematical Association of America.

The July, 1922, number of the AMERICAN JOURNAL OF MATHEMATICS (vol. 44, No. 3) contains the following papers: *Plane involutions of order four*, by T. R. Hollcroft; *Differential equations with a continuous infinitude of variables*, by I. A. Barnett; *The factorization of the rational primes in a cubic domain*, by G. E. Wahlin; *On the structure of finite continuous groups with a single exceptional infinitesimal transformation*, by S. D. Zeldin; *The Laplace-Poisson mixed equation*, by K. P. Williams; *The four color problem*, by Philip Franklin.

In the faculty of sciences of the University of Paris, Dr. E. Cartan, professor of rational mechanics, has been placed in charge of the course in analytic and celestial mechanics, Professor P. Montel of the course in rational mechanics, and Professor A. Denjoy, of the University of Strasbourg, of the course in general mathematics. Dr. Cahen has charge of the conferences in rational mechanics.

The Belgian government has granted a subvention to Professor M. Stuyvaert, of the University of Ghent, for the publication of his *Méthodologie mathématique*.

The Berlin Academy of Sciences has awarded its Steiner prize for a memoir on *The determination of all non-degenerate surfaces of the fifth order on which lie one or more series of conics* to Dr. E. G. Togliatti, of the University of Turin. This subject was announced in 1910 (see this BULLETIN, vol. 17, p. 48), but the award was delayed by war conditions.

The Royal Astronomical Society has awarded its gold medal to Professor A. A. Michelson, of the University of Chicago, for his application of the interferometer to astronomical measurements.

This year the University of Texas inaugurates an annual Research Lectureship to be filled by a member of the faculty selected by the Graduate Council for important achievement in research or for distinguished work in the arts. For 1922–1923 the holder of the Lectureship is Professor M. B. Porter. The subjects and dates of the lectures are *The university and research*, March 28; *Measurable functions*, March 29; *Concepts and problems of mathematics*, March 29; *Integration*, March 30.

A campaign for wider distribution is now being promoted by the journal SCIENTIA, which stands for the philosophic synthesizing of science and the intellectual fraternization of peoples. American subscriptions may be sent to Williams and Wilkins, Baltimore, Md., at ten dollars per year. Among contributors to recent and forthcoming issues are the following Americans: W. D. MacMillan, S. Nearing, L. E. Dickson, W. S. Adams, and R. D. Carmichael.

Professor F. Hartogs, of the University of Munich, has been appointed professor of mathematics at the University of Frankfurt a. M.

Sir W. H. Bragg, of the University of London, has been elected a member of the Paris Academy of Sciences in the section of physics.

Professor A. Sommerfeld, of the University of Munich, delivered a course of eight lectures on atomic structure and theories of radiation at the University of California in February, 1923.

Professor E. G. Bill, of Dartmouth College, has been appointed acting dean during the absence of Dean Laycock.

Professor R. P. Stephens of the University of Georgia has been elected president of the Georgia Academy of Science.

At the Georgia School of Technology, Professor Floyd Field has been elected dean of men, and Mr. H. K. Fulmer (not Fuhner, as in this BULLETIN, July, 1922, p. 324) and Mr. E. R. C. Miles have been appointed instructors in mathematics.

At Mississippi College, Clinton, Mississippi, Mr. J. R. Hitt has been promoted to a full professorship of mathematics.

At St. Ignatius College, Cleveland, Ohio, Prof. E. J. O'Leary has been transferred to St. Louis University, and Mr. H. P. Hecken has been appointed professor of mathematics.

Dr. M. A. Nordgaard, of Columbia University, has been appointed professor of mathematics and head of the department at Antioch College, Yellow Springs, Ohio.

Mr. Daniel Hull, of Notre Dame University, has been promoted to an assistant professorship of mathematics.

Miss Lena R. Cole, of the University of Missouri, has been appointed head of the department of mathematics at Hanover College, Hanover, Indiana.

Mr. Burgoyne Griffing, of the University of Kansas, has been added to the staff of the mathematics department of Des Moines University.

At Beloit College, Beloit, Wisconsin, Professor W. A. Hamilton has been appointed chairman of the Administrative Committee, which is in charge of the college during the interim while a new president is being sought to replace President M. A. Brannon, recently appointed Chancellor of the University of Montana. As this work requires all of Professor Hamilton's time, Professor H. H. Conwell has been appointed acting head of the department of mathematics.

Professor V. H. Wells, of Carleton College, Northfield, Minn., has been granted leave of absence and is spending the year teaching mathematics at Williams College.

NEW PUBLICATIONS

I. PURE MATHEMATICS

ANDOYER (H.). Tables logarithmiques à treize décimales. Paris, Hermann, 1922.

ANER (H.). Algebraisk formteori i förenklad gestalt. Stockholm, A.-B. Svenska Teknologforeningens Forlag, 1921. 8vo. 100 pp.

BIRKEMEIER (W.). Ueber den Bildungswert der Mathematik. Ein Beitrag zur philosophischen Pädogogik. (Wissenschaft und Hypothese, 25.) Leipzig, Teubner, 1923. 191 pp.

BRANDENBERGER (K.). Didaktik des mathematisch-naturwissenschaftlichen Unterrichts. Zürich, Schulthess, 1920. 8vo. 128 pp.

BURALI-FORTI (C.) e MARCOLONGO (R.). Elementi di calcolo vettoriale con numerose applicazioni alla geometria, alla meccanica e alla fisica matematica. 2a edizione, riordenata e ampliata. Bologna, Zanichelli, 1921. 8vo. 19 + 250 pp.

CARNAP (R.). Der Raum. Ein Beitrag zur Wissenschaftslehre. Berlin, Reuther und Reichard, 1922. 87 pp.

ENGEL (F.). See LIE (S.).

FISCHER (P. B.). Determinanten. 2te, verbesserte Auflage. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8vo. 136 pp.

HEIBERG (J. L.). Mathematics and physical science in classical antiquity. Translated by D. C. MacGregor. London, Oxford University Press, 1922. 110 pp.

HJELMSLEV (J.). Elementær Geometri. Tredje Bog. Køpenhavn, Gjellerup, 1921. 8vo. 256 pp.

JUNG (H. W. E.). Einführung in die Theorie der algebraischen Funktionen einer Veränderlichen. Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 8vo. 6 + 246 pp.

KOMMERELL (K.). Der Begriff des Grenzwertes in der Elementarmathematik. Leipzig, Teubner, 1922. 12 + 62 pp.

LIE (S.). Gesammelte mathematische Abhandlungen. Band 3: Abhandlungen zur Theorie der Differentialgleichungen. 1te Abteilung. Herausgegeben von F. Engel. Leipzig, Teubner, 1922.

MACGREGOR (D. C.). See HEIBERG (J. L.).

MARCOLONGO (R.). See BURALI-FORTI (C.).

MOREUX (T.). Pour comprendre la géométrie plane. Paris, Doin, 1922. 252 pp.

PICARD (E.). Traité d'analyse. 3e édition, revue et corrigée. Tome 1. Paris, Gauthier-Villars, 1922. 17 + 594 pp.

STUDY (E.). Die realistische Weltansicht und die Lehre vom Raume. Geometrie, Anschauung und Erfahrung. 2te umgearbeitete Auflage. Teil 1. Braunschweig, Vieweg, 1923. 12 + 85 pp.

WATSON (G. N.). A treatise on the theory of Bessel functions. Cambridge, University Press, 1922. 804 pp.

II. APPLIED MATHEMATICS

ALLIATA (G.). Das Weltbild der Aethermechanik. (Skizze zum Weltbild.) Leipzig, Hillmann, 1922. 8vo. 57 pp.

BECKER (W.). Die Relativitätstheorie gemeinverständlich dargestellt. Leipzig, Hachmeister und Thal, 1921. 8vo. 96 pp.

BEER (F.). Die Einsteinsche Relativitätstheorie und ihr historisches Fundament. 7te Auflage. Wien und Leipzig, Perles, 1922. 8vo. 80 pp.

BROSE (H. L.). See REICHE (F.).

CHARLIER (C. V. L.). Introduction to stellar statistics. Lund, Scientia, 1921. 49 pp.

COHN (E.). Physikalisches über Raum und Zeit. 4te Auflage. Leipzig, Teubner, 1920. 30 pp.

DALLONGEVILLE (E.). Le scintillement de l'énergie. Noyon, E. Dallongeville, 1922.

DOPP (P. H.). See WULF (P. P. T.).

HATFIELD (H. S.). See REICHE (F.).

HOPPE (F.). Elektrische Messinstrumente und Messmethoden. Berlin, R. C. Schmidt, 1922. 8vo. 75 pp.

HORT (W.). Technische Schwingungslehre. Ein Handbuch für Ingenieure, Physiker und Mathematiker bei der Untersuchung der in der Technik angewendeten periodischen Vorgänge. 2te, völlig umgearbeitete Auflage. Berlin, Springer, 1922.

KOHNSTAMM (P.). Lehrbuch der Thermodynamik in ihrer Anwendung auf das Gleichgewicht von Systemen mit gasförmigflüssigen Phasen. Nach Vorlesungen von J. D. van der Waals. 2te, verbesserte Auflage. Teil 1. Leipzig, Barth, 1923. 12 + 287 pp.

LUDEWIG (P.). Radioaktivität. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 133 pp.

MACH (E.). Die Prinzipien der Wärmelehre, historisch-kritisch entwickelt. 4te Auflage. Leipzig, Barth, 1923.

VON MECENSEFFY (E.). Kurvengeometrie des Baumeisters. 6 Hefte. Berlin, W. Ernst, 1920.

POOR (C. L.). Gravitation versus relativity. London, Putnam, 1922. 8vo. 34 + 277 pp.

REICHE (F.). The quantum theory. Translated by H. S. Hatfield and H. L. Brose. New York, Dutton, 1923. 183 pp. $2.50

STICKERS (J.). Die wahre Relativitätstheorie der Physik und die Missbegriffe Einsteins. Bielefeld, W. Breitenbach, 1922. 4 + 57 pp.

THIRRING (H.). Die Idee der Relativitätstheorie. 2te, durchgesehene und verbesserte Auflage. Berlin, Springer, 1922. 4 + 172 pp.

VAN DER WAALS (J. D.). See KOHNSTAMM (R.).

WULF (P. P. T.). La théorie de la relativité d'Einstein. Traduit par P. H. Dopp. Bruxelles, Dewett, 1922. 8vo. 96 pp.

ZACHARIAS (J.). Verborgene Gewalten im Weltgeschehen. Eine neue Raum-Kraft-Lehre. München, O. W. Barth, 1922. 8vo. 91 pp.

JOURNALS OF THIS SOCIETY

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On account of the reduction of stock due to unusual sales, the prices of volumes 13 and 18 of the TRANSACTIONS, and of volume 18 of the BULLETIN have been increased, and volume 11 of the BULLETIN has been withdrawn from sale except with complete sets of volumes 11–26 of the BULLETIN. The present prices, subject to change without further notice, are as follows:

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Princeton Colloquium Lectures, 1909. By G. A. BLISS and EDWARD KASNER. Price, $2.50; to members, $2.00.

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Founded in 1836 by Joseph Liouville

EDITED BY
HENRI VILLAT
WITH THE ASSISTANCE OF
R. DE MONTESSUS DE BALLORE and E. PICARD

This Journal, which has been among the foremost exponents of mathematics for nearly a century, was in danger of discontinuance owing to the changed financial situation caused by the war. By prompt action of a group of French mathematicians, its publication has been assured for the immediate future. This group contains, in addition to the editors, such eminent men as Appell, Borel, Boussinesq, Brillouin, Cartan, Drach, Goursat, Guichard, Hadamard, Koenigs, Lebesgue, Montel, Painlevé, Vessiot.

Papers will be accepted and printed in English as well as in French. Among American contributors to recent or to forthcoming numbers may be mentioned Professors Dickson, Eisenhart, Hancock, Wilczynski. It is hoped that the Journal will become an indispensable element in international mathematical relations.

To insure the permanence of the Journal, subscriptions are invited. The present price is 90 francs, now equivalent to about $6.50. Subscriptions or contributions may be sent to the Editor of this BULLETIN or to H. Villat, 11 rue de Maréchal Pétain, Strasbourg, France.

COLLECTED WORKS OF LEONARD EULER

On the two-hundredth anniversary of the birth of Euler, a committee of the *Society of Swiss Naturalists* launched the project of international cooperation for the publication of his collected works. Academies and individuals subscribed for about 300 sets, and about one hundred thousand (100,000) Swiss francs were collected, for the most part in Switzerland; the American Mathematical Society subscribed five thousand (5,000) francs. Eighteen (18) of the estimated seventy (70) volumes have been published.

By reason of the European War nearly one-half of the subscribers have been unable to meet their obligations in full. Under these circumstances, a considerable number of new subscribers must be secured if the completion of the undertaking is to be possible in the near future. There are three plans for subscribing. Those libraries or individuals wishing for information with a view to joining in promoting this great international undertaking, should communicate with the Official Representative of the Euler Committee for the United States and Canada,

R. C. ARCHIBALD,
Brown University,
Providence, R. I.

ANNALI DI MATEMATICA PURA ED APPLICATA

This important Italian mathematical periodical is in straitened financial circumstances, owing to conditions arising from the war. Subscriptions for its support are suggested. First-class manuscripts in English, French, or Italian will be accepted. The price per subscription is forty Lire per volume of four numbers, of about eighty pages each; ordinarily a number appears each quarter. Subscriptions and cash gifts will be received by **Professor Virgil Snyder, 214 University Avenue, Ithaca, New York.**

OFFICIAL COMMUNICATIONS

The Council announces that Professor J. L. Coolidge has accepted the chairmanship of the Committee on Endowment.

Professor E. V. Huntington has been appointed a representative of the Society in the Division of Physical Sciences of the National Research Council for a period of three years beginning July 1, 1923.

Arrangements have been made to have the 1923 volume of the Transactions printed by the firm of Lütke und Wulff of Hamburg.

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, April 28, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than April 14.

THE SUMMER MEETING OF THE SOCIETY, AT VASSAR COLLEGE, September 6–8, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than August 8.

NEW YORK CITY, October 27, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than October 13.

R. G. D. RICHARDSON,
Secretary of the Society.

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Whole Number 312

CONTENTS

PAGE

The Twenty-ninth Annual Meeting of the Society. By R. G. D. RICHARDSON 97

The December Meeting of the Society in Evanston. By ARNOLD DRESDEN 117

The November Meeting of the Southwestern Section. By E. B. STOUFFER 125

A Set of Axioms for Line Geometry. By M. G. GABA 128

Shorter Notices 139

Notes 141

New Publications 143

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